MODERN ALGEBRA 1: HOMEWORK 2

The problem numbers refer to Artin's *Algebra* (2nd edition).

(1) Chapter 1: 1.7

(2) Chapter 1: 4.4

(3) Chapter 1: 6.2 (*Hint: Use cofactors – Theorem 1.6.9*).

(4) Chapter 2: 2.4

(5) Suppose $(ab)^2 = a^2b^2$ for all a, b in a group G. Prove that G is abelian.

(6) Prove that the set of 3×3 matrices of the form

$$\begin{pmatrix} 1 & a & b \\ & 1 & c \\ & & 1 \end{pmatrix}, \quad a, b, c \in \mathbf{R}$$

is a subgroup of $\mathrm{GL}_3(\mathbf{R})$. It is called the *Heisenberg group*.

(7) Recall that the Klein four group consists of the matrices

$$\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}.$$

Find all the subgroups of the Klein four group.

(8) Suppose G is a finite group whose order is even. Show that there exists an element of G different from the identity which is its own inverse.

(9) Recall that for an element $x \in G$, the subgroup generated by x is the subgroup

$$\{\ldots, x^{-2}, x^{-1}, e, x^1, x^2, \ldots\}.$$

Let $n \geq 1$ be an integer. Write a 2×2 matrix that generates a subgroup of order n of $\mathrm{GL}_2(\mathbf{R})$. Write a matrix that generates an infinite subgroup of $\mathrm{GL}_2(\mathbf{R})$. Justify (i.e. prove) your answers.