MODERN ALGEBRA 1: HOMEWORK 3

(1) Chapter 2: 4.1

(2) In the following problem, G is a group. All small letters stand for elements of G.
  (a) Show that $\langle x \rangle = \langle x^{-1} \rangle$. Conclude that x and x^{-1} have the same order.
  (b) Show that x and yxy^{-1} have the same order. Use this to show that ab and ba have the same order.
  (c) Let H be a group and $\phi\colon G \to H$ an isomorphism. Show that x and $\phi(x)$ have the same order.

(3) Let G be a cyclic group of order n, and let $x \in G$ be a generator. Let $H \subset G$ be a subgroup. By considering the set $S \subset \mathbf{Z}$ defined by

$$S = \{i \in \mathbf{Z} \mid x^i \in H\},$$

solve the following.
  (a) Show that $H = \langle x^d \rangle$ for some d that divides n. What is the order of H?
  (b) Let a be any integer (not necessarily dividing n). What is the order of x^a?
  (c) Conclude that all subgroups of G are cyclic, their order divides n, and moreover, for every divisor d of n, there is a unique subgroup of G of order d.

This is a step-by-step and expanded version of Chapter 2: 4.5.
Hint: Start by showing that S is a subgroup of $\mathbf{Z}^+$.

(4) A *transposition* is a permutation that switches two elements and fixes the others. In cycle notation, it corresponds to (ab), where $a, b \in \{1, \ldots, n\}$ are distinct.
  (a) Show that every permutation $p \in \mathbf{S}_n$ can be expressed as a product of transpositions. (*Hint: Use induction on n*).
  (b) Give an example showing that such an expression is not unique.

(5) Chapter 2: 4.9

(6) Chapter 2: 4.10

(7) (a) Show that $\mathbf{S}_3$ is not isomorphic to $\mathbf{Z}_6$.
  (b) Show that $\mathbf{Z}_6$ is isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_3$

(8) Chapter 2: 6.10 (a) or (b) – do whichever you like.