# MODERN ALGEBRA 1: HOMEWORK 5

(1) Chapter 2: 7.1

(2) Chapter 2: 8.1

(3) Let $G$ be a group of order $p$, where $p$ is a prime. Show that $G$ must be isomorphic to $\mathbf{Z}_p$.

(4) Chapter 2: 8.10

(5) Suppose $H \subset G$ is a subgroup such that $aHa^{-1} \subset H$ for all $a \in G$. Show that $H$ is normal. (This is a trick we used in class.)

(6) Give an example of a group $G$ and a normal subgroup $H \triangleleft G$ such that $G$ is not isomorphic to $H \times (G/H)$. Give an example where $H$ and $G/H$ are abelian, but $G$ is not. (It is OK to kill two birds with one stone.)
Remember this example for the future.

(7) The *center* $Z(G)$ of a group $G$ is the subset of elements that commute with every other element:

$$Z(G) = \{g \in G \mid gx = xg \text{ for all } x \in G.\}.$$

Show that $Z(G)$ is a normal subgroup of $G$.

(8) Recall the group of complex numbers of absolute value 1

$$U = \{z \in \mathbf{C} \mid |z| = 1\},$$

where the group operation is multiplication.
(a) Show that $\mathbf{R}^+/\mathbf{Z}^+$ is isomorphic to $U$. (*Hint: An isomorphism can be constructed using exponentiation.*)
(b) Find one element of finite order and one element of infinite order in $\mathbf{R}^+/\mathbf{Z}^+$.