MODERN ALGEBRA 1: HOMEWORK 6

(1) Chapter 2: 8.3

(2) Chapter 2: 8.6

(3) Chapter 2: 12.5

(4) Let $H \subset G$ be a subgroup. The *normalizer* of H, denoted by $N(H)$, is defined by

$$N(H) = \{g \in G \mid gHg^{-1} = H\}.$$

(a) Show that $N(H)$ is a subgroup of G and $N(H) = G$ if and only if $H \triangleleft G$.
(b) Show that H is a normal subgroup of $N(H)$.

Remark. Keep in mind that saying $gHg^{-1} = H$ is *not* the same as saying $ghg^{-1} = h$ for every $h \in H$. By $gHg^{-1} = H$ we mean that the collection $\{ghg^{-1} | h \in H\}$ as a whole is the same as the collection $\{h \mid h \in H\}$.

(5) Let $G = \mathrm{GL}_2(\mathbf{R})$ and $H \subset G$ the subgroup of diagonal matrices, namely matrices of the form

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad ab \neq 0.$$

Find $N(H)$ and identify the quotient group $N(H)/H$.
(Optional) Generalize the previous result to $\mathrm{GL}_n(\mathbf{R})$.

(6) Let $Q = \{\pm\mathbf{1}, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$ be the quaternion group from page 47 of the book. Find all homomorphisms from $\mathbf{Z}_2$ to Q and from $\mathbf{Z}_4$ to Q. Are there any nontrivial homomorphisms from $\mathbf{Z}_3$ to Q?

(7) Find all subgroups of Q. For a slick solution, proceed as follows.
(a) Show first that all nontrivial subgroups must contain $\{\pm 1\}$.

(b) Check that $\{\pm 1\} \triangleleft Q$. Identify the quotient $Q/\{\pm 1\}$ and use the correspondence theorem for subgroups.

Observe that all subgroups of Q are normal subgroups, although Q is not abelian!