MODERN ALGEBRA 2: PRACTICE PROBLEMS FOR THE FINAL

Use the following practice problems to test your understanding of the last third of the course, namely field extensions and Galois theory.

(1) Let $\mathbf{F}_p \subset K$ be an extension of degree n. Show that all irreducible polynomials of degree n in $\mathbf{F}_p[x]$ have a root in K. In fact, show that all irreducible polynomials of degree n have n distinct roots in K. If α is one root, how will you obtain all the other roots?

(2) For the following polynomials, describe the splitting field, the Galois group, and the action of the Galois group on the roots.
   (a) $x^3 - 2$ over $\mathbf{Q}$.
   (b) $x^3 - 2$ over $\mathbf{Q}(\omega)$.
   (c) $x^4 + 1$ over $\mathbf{Q}$.
   (d) $x^4 + 1$ over $\mathbf{F}_3$.

(3) Describe the splitting field of $(x^2 - 2x - 1)(x^2 - 2x - 7)$ over $\mathbf{Q}$. How many subfields does it have? How many of those are Galois over $\mathbf{Q}$?

(4) Determine the splitting field of $x^3 + x + 1$ over $\mathbf{Q}$. Does this field have a subfield which has degree 2 over $\mathbf{Q}$? If it does, identify it. Otherwise, explain why not.

(5) Find the splitting field and the Galois group of $x^4 - 8x^2 + 11$. Use the action of the Galois group on the roots to show that $x^4 - 8x^2 + 11$ is irreducible over $\mathbf{Q}$.

(6) Let $F = \mathbf{Q}(\omega)$. Determine the Galois group over F of the splitting field of (a) $\sqrt[3]{2 + \sqrt{2}}$ (b) $\sqrt{2 + \sqrt[3]{2}}$.

(7) Let $p(x)$ be a real polynomial whose discriminant is positive. Show that $p(x)$ must have an even number of pairs of non-real complex roots.

(8) Let $F \subset K$ be a Galois extension and $\alpha \in K$. Let $\{\alpha_1, \ldots, \alpha_m\}$ be the orbit of α under the Galois group. Show that $(x - \alpha_1) \ldots (x - \alpha_m)$ is an irreducible polynomial in $F[x]$.

(9) Let $\zeta = e^{2\pi i/7}$. Find a generator of the Galois group of $\mathbf{Q}(\zeta)/\mathbf{Q}$. Use it to determine the degrees of the following elements over $\mathbf{Q}$: (a) $\zeta + \zeta^5$ (b) $\zeta^3 + \zeta^5 + \zeta^6$.

(10) Let $p(x)$ be an irreducible quartic polynomial over $\mathbf{Q}$ whose resolvent cubic has all three rational roots. Show that the Galois group of $p(x)$ is the Klein four group.

(11) Let $p(x) \in \mathbf{Q}[x]$ be a polynomial whose Galois group is a Dihedral group. Show that the roots of $p(x)$ can be expressed using radicals and roots of unity.