MODERN ALGEBRA 2: HOMEWORK 10

(1) Find the Galois groups (over $\mathbf{Q}$) of the following two cubics: $x^3 - 3x^2 + 1$, and $x^3 + x^2 - 2x + 1$.

(2) Let $\mathbf{Q} \subset K$ be the splitting field of $x^3 - 3x + 1$. Show that $\mathrm{Gal}(K/\mathbf{Q}) = \mathbf{Z}/3\mathbf{Z}$ but K cannot be obtained by adjoining a cube-root to $\mathbf{Q}$. (That is, $K \neq \mathbf{Q}(a)$ for any $a \in K$ with $a^3 \in \mathbf{Q}$.)

(3) Let $p(x) = x^3 - 2x + 2$. Use symmetric functions to find the monic polynomial whose roots are the squares of the roots of $p(x)$.

(4) Let $p(x) \in \mathbf{Q}[x]$ be an irreducible monic quartic polynomial with (possibly complex) roots $\alpha_1, \ldots, \alpha_4$. Set

$$\beta_1 = \alpha_1\alpha_2 + \alpha_3\alpha_4, \quad \beta_2 = \alpha_1\alpha_3 + \alpha_2\alpha_4, \quad \beta_3 = \alpha_1\alpha_4 + \alpha_2\alpha_3.$$

Show that $r(x) = (x - \beta_1)(x - \beta_2)(x - \beta_3)$ has coefficients in $\mathbf{Q}$. The polynomial $r(x)$ is called the *resolvent cubic* of $p(x)$.

(5) Let $p(x) \in \mathbf{Q}[x]$ be an irreducible monic quartic polynomial whose resolvent cubic $r(x) \in \mathbf{Q}[x]$ is irreducible. Show that the Galois group of $p(x)$ is either A_4 or S_4. You can use that the only transitive subgroups of S_4 are D_2, C_4, D_4, A_4, and S_4. Use this criterion and the discriminant to exhibit quartic polynomials with Galois groups A_4 and S_4.

(6) Show that all the possibilities listed above, namely D_2, C_4, D_4, A_4, and S_4, can arise as Galois groups of quartic polynomials. Feel free to refer to any previous problems or statements from class to support your claims.

Determining whether a given finite group arises as the Galois group of a polynomial in $\mathbf{Q}[x]$ is a hard unsolved problem called the *inverse Galois problem*.

Remark. Some of the problems above may require you to express a symmetric polynomial in terms of the elementary symmetric polynomials. Although I encourage you to do a few such examples by hand, this could get quite tedious. The following functions from `Mathematica` (which also work on WolframAlpha) can help:

- `SymmetricReduction[f,{`$x_1, \ldots, x_n$`}]`: Expresses the given symmetric function f of $x_1, \ldots, x_n$ in terms of the elementary symmetric functions.
- `SymmetricReduction[f,{`$x_1, \ldots, x_n$`},{`$s_1, \ldots, s_n$`}]`: First expresses the given symmetric function f of $x_1, \ldots, x_n$ in terms of the elementary symmetric functions and then evaluates the resulting expression at the given values of $s_1, \ldots, s_n$.