MODERN ALGEBRA 2: HOMEWORK 2

(1) Let F be a field. Show that a polynomial $p(x) \in F[x]$ of degree n has at most n roots in F.

(2) Let R be a ring. The whole ring R is an ideal of itself, called the *unit ideal*. Show that if an ideal I contains a unit, then it is the unit ideal.

(3) Let R be a ring and let $a, b \in R$. Show that $(a) = (b)$ if and only if $a = ub$ for some unit $u \in R$.

Remark. *As stated, this statement is wrong. It is true if R is an integral domain. For a counterexample, consider* $R = \mathbf{C}[x, y, z]/((1 - xy)z)$. *Then we have* $(z) = (yz)$. *Indeed, it is clear that* $(yz) \subset (z)$, *and* $z = xyz$ *implies that* $(z) \subset (yz)$. *However, it is not true that there is a unit* u *such that* $yz = uz$.

(4) Every non-zero ring has at least two ideals, the zero ideal and the unit ideal. Show that a non-zero ring is a field if and only if it has no other ideals.

(5) Show that the characteristic of a field is a prime number.

(6) §11.3: 3.12 (Sum of two ideals)

(7) §11.4: 4.3 (Identify some quotient rings)

(8) §11.4: 4.4 (Are $\mathbf{Z}[x]/(x^2 + 7)$ and $\mathbf{Z}[x]/(2x^2 + 7)$ isomorphic?)

(9) §11.5: 5.2 (Adjoining a pre-existing element)