MODERN ALGEBRA 2: HOMEWORK 5

Hint (General suggestion). Remember that if $f(x) \in \mathbf{Z}[x]$ is irreducible in $\mathbf{Z}[x]$ then it is also irreducible in $\mathbf{Q}[x]$. In turn, to show $f(x)$ is irreducible in $\mathbf{Z}[x]$ you can use the information gained from reducing modulo p.

(1) Chapter 12, §4.1
Hint: Here is a very slick way of doing this problem. Let us take part (b), for example. Show that *any* irreducible polynomial $p(x)$ in $\mathbf{F}_2[x]$ of degree 1, 2, or 4 must divide $x^{16} - x$ by showing that $x^{16} = x$ in $\mathbf{F}_2[x]/(p(x))$. To show $x^{16} = x$, it is helpful to consider the group $\mathbf{F}_2[x]/(p(x)) \setminus \{0\}$ under multiplication.

(2) Chapter 12, §4.3

(3) Chapter 12, §4.5

(4) ~~Chapter 12, §4.12~~ (Skip)

(5) Chapter 12, §4.16

(6) Chapter 15, §1.2

(7) Chapter 15, §2.1

(8) Show that there exist real numbers that are transcendental over $\mathbf{Q}$ by showing that the set of real numbers algebraic over $\mathbf{Q}$ is countable.
Remember that a set S is *countable* if there is a bijection between $\mathbf{Z}$ and S. Let us say that a set is *at most countable* if it is finite or countable. You may freely use the following facts about countable sets:
(a) $\mathbf{Q}$ is countable.
(b) Subsets of countable sets are at most countable.
(c) Images of countable sets are at most countable.
(d) An (at most) countable union of at most countable sets is at most countable.
(e) $\mathbf{R}$ is not countable.