MODERN ALGEBRA 2: HOMEWORK 8

SPLITTING FIELDS AND GALOIS GROUPS

(1) Determie the degrees of the splitting fields of the following polynomials over $\mathbf{Q}$: (a) $x^4 - 1$ (b) $x^4 + 1$.
(2) Let $\omega = e^{2\pi i/3}$. Show that the extension $\mathbf{Q} \subset \mathbf{Q}(\omega, \sqrt[3]{2})$ is Galois and its Galois group is isomorphic to S_3.
(3) Let $F \subset K$ be a splitting field of $p(x) \in F[x]$ and set $n = \deg(p(x))$. Show that $\mathrm{Gal}(K/F)$ is a subgroup of S_n.

FINITE FIELDS AND GALOIS GROUPS

(4) Let $F \subset K$ be finite fields where $|F| = p^m$ and $|K| = p^n$. Show that m divides n.
(5) Conversely, show that if m divides n, then the subset

$$\{x \in K \mid x^{p^m} = x\} \subset K$$

is a subfield of order p^m.
(6) Let $F \subset K$ be finite fields where $|F| = p^m$ and $|K| = p^n$. Show that K/F is Galois, and $\mathrm{Gal}(K/F)$ is cyclic of order n/m, generated by the automorphism

$$x \mapsto x^{p^m}.$$

SEPARABILITY AND PERFECT FIELDS

(7) Let F be a field of characteristic p and $f(x) \in F[x]$ a polynomial. Show that $Df(x) = 0$ if and only if $f(x) = g(x^p)$ for some polynomial $g(x) \in F[x]$.
(8) F called *perfect* if the Frobenius homomorphism $F \to F$ given by $x \to x^p$ is an isomorphism. Show that a finite field is perfect.
(9) Let F be a perfect field and $f(x) \in F[x]$ an irreducible polynomial. Show that $f(x)$ is separable.