MODERN ALGEBRA 2: HOMEWORK 9

Note: The Galois group of a polynomial means the Galois group of its splitting field.

(1) Use Galois theory to show that $5^{1/3}$ does not lie in $\mathbf{Q}(2^{1/3})$.
Hint: Work in $\mathbf{Q}(\omega, 2^{1/3})$.

(2) Let p be a prime. Show that the Galois group of $x^p - 2$ is isomorphic to the group of matrices

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix},$$

where $a, b \in \mathbf{F}_p$ and $a \neq 0$.

(3) Suppose $f(x) \in \mathbf{Q}[x]$ is an irreducible quartic whose Galois group is S_4. Let α be a root of $f(x)$, and let $K = \mathbf{Q}(\alpha)$. Show that $K/\mathbf{Q}$ is an extension of degree 4 and K has no subfields other than K and $\mathbf{Q}$. (Among other things, this gives many examples of numbers of degree four that are not constructible.)

(4) Suppose $f(x) \in \mathbf{Q}[x]$ is a cubic whose Galois group is cyclic of order 3. Show that all roots of $f(x)$ must be real.

(5) (Biquadratic extensions) Let $K = \mathbf{Q}(\sqrt{a}, \sqrt{b})$, where $a, b \in \mathbf{Q}$ and $\sqrt{a}, \sqrt{b} \notin \mathbf{Q}$. Show that $K/\mathbf{Q}$ is Galois with Galois group $\mathbf{Z}/2\mathbf{Z}$ or $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$. Conversely, show that any Galois extension $K/\mathbf{Q}$ with Galois group $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ has the form $K = \mathbf{Q}(\sqrt{a}, \sqrt{b})$.

(6) Let $\zeta_p = e^{2\pi i/p}$. Show that $\mathbf{Q}(\zeta_p)$ contains a unique quadratic extension of $\mathbf{Q}$.

(7) Let $K = \mathbf{Q}(i, 2^{1/4})$. Show that $K/\mathbf{Q}$ is Galois and is Galois group is D_4 (the Dihedral group with 8 elements).