ANALYSIS AND OPTIMIZATION: FINAL EXAM PRACTICE PROBLEMS

SPRING 2016

1. Review

The final exam will be cumulative, but with a bias towards the part of the course covered after the second midterm. For your review, here is a list of topics that we have covered in the entire course, not necessarily in the chronological order.

- Linear programming
  - Linear optimization in two variables using graphs.
  - The simplex method.
  - Duality and shadow prices.

- Linear algebra/quadratic forms
  - Quadratic forms, their definiteness, eigenvalues, and diagonalization (spectral theorem).
  - Criteria for definiteness using principal minors.
  - Quadratic forms with linear constrains.

- Basic topology
  - Open and closed sets.
  - Bounded, compact sets, and the maximum theorem.
  - Convex sets.

- Basic analysis of functions of several variables
  - Gradients, stationary points.
  - Taylor approximation.
  - Hessians and the criteria for local min, local max, saddle points.
  - The implicit function theorem.
  - Convex/concave functions and their properties.

- Optimization
  - Unconstrained optimization using stationary points. Implications of convexity/concavity.
  - Constrained optimization using Lagrange multipliers, second order criteria using bordered Hessians.
  - The Lagrangian function and implications of its convexity/concavity.
  - Inequality constraints and KKT conditions.
  - The envelope theorem.

- Calculus of variations
  - The Euler–Lagrange equation.
  - Solving the Euler–Lagrange equation in simple cases.

2. Practice problems

These are practice problems for the part of the course after the second midterm. If you get unworkably ugly solutions, please let me know.

(1) Find all points (x, y, z) that satisfy the first order conditions for local optima for $x + y + z$ subject to $x^2 + y^2 + z^2 = 1$ and $x - y - z = 1$. Classify them as local maxima, local minima, or saddle points.

(2) Minimize the function $xy + x^2$ subject to $x^2 + y \leq 2$ and $y \geq 0$.

(3) Find the global minimum and maximum value of the function xz on the set defined by $x + y - z = 0$, $x^2 + y \leq 2$, and $y \geq 1$.

(4) Consider the function $2x^2 + 4y^2$ on the set $x^2 + y^2 = 1$. Use Lagrange multipliers to find the global minimum and maximum of this function. What do the the second order criteria say at $(1, 0)$?

(5) Maximize $x + 2y$ subject to $3x^2 + y^2 \leq 1$, $x - y \leq 1$, $x \geq 0$, and $y \geq 0$.

(6) Let c be a positive constant. Consider the problem of maximizing $\ln(x+1)+\ln(y+1)$ subject to $x + 2y \leq c$ and $x + y \leq 2$. Let $V(c)$ be the maximum value.
   (a) Find $V(5/2)$ and the (x, y) that achieve this value.
   (b) Find $V'(5/2)$.

(7) A spring of natural length L extended to length $L + x$ contains energy $\frac{1}{2}kx^2$, where k is a constant called the *stiffness* of the spring. Suppose n springs of natural lengths $L_1, \ldots, L_n$ and stiffnesses $k_1, \ldots, k_n$ are stringed together and the resulting contraption is extended to length $L_1 + \cdots + L_n + \ell$.
   (a) Find the extensions of the individual strings, assuming that the system minimizes the total energy. Justify why the solution you found is a global minimum.
   (b) Find the rate of change of the energy of the contraption with respect to k_i and L_i using the envelope theorem.

(8) Use an appropriate bordered matrix to show that the quadratic form $-5x^2 + 2xy + 4xz - y^2 - 2z^2$ is negative definite on the subspace of $\mathbf{R}^3$ defined by $x + y + z = 0$ and $4x - 2y + z = 0$.

(9) Consider a closed box with sides x, y, z and fixed volume V. Set up the Lagrange multiplier problem to minimize the surface area, find the candidate solution(s), and find the global minimum.

(10) Find the point on the line $x = y$ closest to the circle of radius 1 and center $(5, 2)$ using Lagrange multipliers. Make sure that your answer makes geometric sense. What is the approximate change in the minimum distance if the center of the circle is moved from $(5, 2)$ to $(5 + \epsilon, 2 - \epsilon)$?

(11) Minimize

$$\int_0^1 x^2 + 2tx\dot{x} + \dot{x}^2 \, dt,$$

subject to $x(0) = 1$ and $x(1) = 2$. You may assume that a minimum exists.

(12) The discounted total utility function for an investment strategy $K(t)$ over a period T is given by

$$\int_0^T e^{-t/4} \ln(2K - \dot{K}) \, dt.$$

Find a function $K(t)$ that maximizes this subject to $K(0) = K_0$ and $K(T) = K_T$. You may assume that a maximum exists.

(13) By solving an Euler–Lagrange equation, find the curve of length π joining $(0,0)$ and $(1,0)$ that together with the straight line from $(0,0)$ to $(1,0)$ encloses the maximum area.

Hint: This is a calculus of variations problem with a constraint.