ANALYSIS AND OPTIMIZATION: HOMEWORK 2

SPRING 2016

Due date: Wednesday, February 10.

(1) Find the rank of the matrix

$$\begin{pmatrix} 1 & 0 & -1 & 2 \\ -2 & 1 & 1 & -1 \\ -3 & 2 & 1 & 0 \end{pmatrix}.$$

(2) Let A be an invertible $n \times n$ matrix and let $v_1, \ldots, v_k \in \mathbf{R}^n$ be linearly independent vectors. Show that $Av_1, \ldots, Av_k \in \mathbf{R}^n$ are also linearly independent.

(3) Using Gaussian elimination, find all solutions to the linear system

$$x - y + z = 2, \quad x + 2y - z = 3, \quad 2x + y + 3z = 21.$$

How many degrees of freedom are there?

(4) Show that the following set is closed in $\mathbf{R}^2$.

$$S = \{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 \geq \sin(x + y),\ x \leq y, \text{ and } x^2 + y^2 = 10xy\}.$$

(5) Give examples of subsets $S \subset \mathbf{R}$ and continuous functions f such that
  (a) S is closed but $f(S)$ is not closed ($f : S \to \mathbf{R}$)
  (b) S is open but $f(S)$ is not open ($f : S \to \mathbf{R}$),
  (c) S is bounded but $f(S)$ is not bounded ($f : S \to \mathbf{R}$),
  (d) S is compact but $f^{-1}(S)$ is not compact ($f : \mathbf{R} \to \mathbf{R}$).

(6) Show that an extreme point of a convex set must be a boundary point of that set. Is the converse true?

(7) Which of the following sets are open or closed? To justify your answers, write down the set of boundary points and the set of interior points in each case.
  (a) $(-\infty, 2] \cup \{5\}$ in $\mathbf{R}$
  (b) The set of integers $\mathbf{Z}$ in $\mathbf{R}$.
  (c) $\{\vec{x} \in \mathbf{R}^n \mid 4 \leq |\vec{x}| \leq 7\}$ in $\mathbf{R}^n$.
  (d) $\{(x, y) \in \mathbf{R}^2 \mid e^{x+y} \in (1, 5)\}$ in $\mathbf{R}^2$.
  (e) $\left\{\frac{1}{n} \mid n = 1, 2, \ldots\right\}$ in $\mathbf{R}$.

(8) Let $S \subset \mathbf{R}^n$ be convex, and $\vec{x}, \vec{y}, \vec{z} \in S$. Let $a, b, c \in \mathbf{R}$ be such that $0 \leq a, b, c \leq 1$ and $a + b + c = 1$. Show that $a\vec{x} + b\vec{y} + c\vec{z} \in S$.
Extra: Can you think (and prove) a generalization for an arbitrary number of vectors?.