ANALYSIS AND OPTIMIZATION: HOMEWORK 5

SPRING 2016

Due date: Wednesday, March 9.

(1) Write the symmetric matrix associated with the quadratic form

$$3x_1^2 - 2x_1x_2 + 4x_1x_3 + 8x_1x_4 + x_2^2 + 3x_2x_3 + x_3^2 - 2x_3x_4 + x_4^2.$$

Write the quadratic form associated with the symmetric matrix

$$\begin{pmatrix} 1 & -3 & 5 \\ -3 & -2 & 0 \\ 5 & 0 & 9 \end{pmatrix}.$$

(2) Determine if the following quadratic forms are positive (semi) definite, negative (semi) definite, or indefinite.

(a) $-x_1^2 + 2x_1x_2 - 6x_2^2$.

(b) $4x_1^2 + 2x_1x_2 + 25x_2^2$.

(c) $3x_1^2 - 2x_1x_2 + 3x_1x_3 + x_2^2 + 3x_3^2$.

(3) Consider the quadratic form

$$-x_1^2 + 6x_1x_2 - 9x_2^2 - 2x_3^2.$$

Find the associated matrix A. Also find a diagonal matrix D and an orthogonal matrix P such that

$$P^T A P = D.$$

(4) For which values of c is the quadratic form

$$3x^2 - (5+c)xy + 2cy^2$$

(a) positive definite, (b) positive semidefinite, (c) indefinite?

(5) Let B be any $n \times n$ matrix and let $A = B^T B$. Show that A is symmetric and the quadratic form associated with A is positive semidefinite. If B is invertible, show that the quadratic form is actually positive definite.

(6) Suppose $q(\vec{x})$ is a positive semidefinite quadratic form. Use the spectral theorem to show that there exist linear functions $L_1(\vec{x}), \dots, L_n(\vec{x})$ such that

$$q(\vec{x}) = L_1(\vec{x})^2 + \cdots + L_n(\vec{x})^2.$$

Remark: The analogous question for higher degree polynomials—when can polynomials that only take non-negative values be written as sums of squares?—has been and continues to be a topic of research. See the article *Sums of Squares* by Olga Taussky (*Mathematical Association of America, 1970*) if you are interested.