ANALYSIS AND OPTIMIZATION: HOMEWORK 7

SPRING 2016

Due date: Wednesday, April 13.

(1) Maximize $e^x + y + z$ subject to $x + y + z = 1$ and $x^2 + y^2 + z^2 = 1$. Estimate the maximum value when the constraints are changed to $x + y + z = 1.02$ and $x^2 + y^2 + z^2 = 0.98$.

(2) Let m be a positive real number. Maximize $1 - x^2 - y^2$ subject to $x + y = m$ using Lagrange multipliers. You may assume that a maximum exists.

Now think of the maximum value as a function of m, and find its rate of change with respect to m. Verify that the rate of change is equal to the Lagrange multiplier.

(3) Find the four points that satisfy the first order Lagrange multiplier conditions for the problem of maximizing $x^2 + y^2$ subject to $2x^2 + y^2 = 2$. Using the second order criterion, classify the four points as local maxima, local minima, or saddle points.

(4) Do the same for the problem of maximizing $x + y + z$ subject to $x^2 + y^2 + z^2 = 1$ and $x - y - z = 1$.

(5) Maximize xy subject to $x + y^2 \leq 2$, $x \geq 0$, and $y \geq 0$ using Karush–Kuhn–Tucker conditions.
Hint: Begin by consider the two cases $x + y^2 < 2$ or $x + y^2 = 2$.

(6) Let $S = \{(x, y) \mid y \geq e^x, y \geq e^{-x}\}$. Check whether the two constraints satisfy constraint qualification at $(0, 1)$. Sketch the region S showing the point $(0, 1)$. Suppose f is a function that attains its maximum on S at $(0, 1)$. Thinking of $\nabla f(0, 1)$ as a vector with its tail end at $(0, 1)$, show the possible locations of its tip (using KKT).

(7) Let $Q(\vec{x})$ be a positive definite quadratic form with associated symmetric matrix A. Set $S = \{\vec{x} \mid Q(\vec{x}) = 1\}$. Show that S is compact. Using Lagrange multipliers, show that the point on S that is closest to the origin is an eigenvector of A and its eigenvalue is the largest of the eigenvalues of A.