ANALYSIS AND OPTIMIZATION: HOMEWORK 9

SPRING 2016

Due date: Wednesday, April 27.

(1) Write down the Euler–Lagrange equation associated with the following optimization problems
   (a) $\int x^2 + \dot{x}^2 + 2xe^t \; dt$
   (b) $\int -e^{\dot{x}-x} \; dt$

(2) Consider the problem of finding a curve of shortest length in the (x,t)-plane that joins $(0,0)$ to (a,b). Formulate this as a calculus of variations problem, find the associated Euler–Lagrange equation, and check that the straight line satisfies the equation.

(3) Minimize $\int_1^2 \frac{\dot{x}^2}{t^2} \; dt$ subject to $x(1) = 0$ and $x(2) = 1$.

(4) Suppose an investment amount K yields returns at rate cK, which can be consumed immediately or re-invested. Let C be the rate of consumption and R the rate of re-investment. Then we have the equations $cK = C + R$, and $R = \dot{K}$ from which we get $C = cK - \dot{K}$. Suppose consumption at rate C at time t gives utility $e^{-rt} \ln C$.
   (a) Which function $K(t)$ maximizes the total utility
$$\int_0^1 e^{-rt} \ln C \; dt$$
   subject to $K(0) = K_0$ and $K(1) = 0$? Here $r > 0$ and $c > 0$ are constants.
   (b) Using software of your choice, sketch the graph of the consumption function $C(t)$ for r very small and r very large (take $c = 0.1$ and $K_0 = 1$).

   Remark. The factor e^{-rt} in front of the utility function is common in economics and is called a *discounting factor*. It results in greater weight given to immediate utility compared to future utility. Larger values of r mean greater preference for immediate utility.

   Hint: A general solution to the differential equation $\ddot{K} - (\alpha + \beta)\dot{K} + \alpha\beta K = 0$ is given by $K(t) = Ae^{\alpha t} + Be^{\beta t}$.

(5) One formulation of the laws of physics says that the path taken by an object minimizes the integral $\int (T - V) \; dt$, where T is the kinetic energy and V is the potential energy. Suppose an object is falling vertically and its height from the ground at time t is $h(t)$. Write the Euler–Lagrange equation to minimize $\int (T - V) \; dt$ and check that it is equivalent to Newton's second law ($F = ma$).
   Hint: The potential energy at height h is mgh and the kinetic energy at speed v is $\frac{1}{2}mv^2$.