ANALYSIS AND OPTIMIZATION: MIDTERM 2 PRACTICE PROBLEMS

SPRING 2016

1. Review

For your review, here is a list of topics that we have covered.

- Gradients and stationary points.
- Taylor approximation.
- Quadratic forms and their definite-ness. Relationship with eigenvalues and the spectral theorem. Criteria using principal minors.
- Second order criteria to determine local min/max for a stationary point.
- Convex and concave functions. Relationship with the first derivatives (the gradient). Relationship with the second derivatives (the Hessian). Behavior under sums and compositions. Jensen's inequality.
- The implicit function theorem.
- Lagrange multipliers for constrained optimization.

2. Practice problems

(1) Write the definition of a convex function. Let $f(\vec{x})$ and $g(\vec{x})$ be two convex functions on $\mathbf{R}^n$. Using the definition, show that the function $h(\vec{x})$ defined by

$$h(\vec{x}) = \max(f(\vec{x}), g(\vec{x}))$$

is also convex.

(2) Use Jensen's inequality to prove that for positive real numbers $x_1, \ldots, x_n$, we have

$$\sqrt[3]{\frac{x_1^3 + \cdots + x_n^3}{n}} \geq \frac{x_1 + \cdots + x_n}{n}.$$

(3) Find the global minimum and maximum of the function $f(x, y) = 2x^3 + 4y^3$ on the set $S = \{x^2 + y^2 \leq 1\}$ by the following outline.

(a) Show that the maximum and the minimum exists.

(b) Using the gradient, find the possible points where the max/min could be achieved on the interior $\{x^2 + y^2 < 1\}$.

(c) Using Lagrange multipliers, find the possible points where the max/min could be achieved on the boundary $\{x^2 + y^2 = 1\}$.

(d) Check all the possibilities.

(4) Let A be the matrix

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

(a) Write down the quadratic form $Q(x, y, z)$ associated with A.
(b) Show that the function $f(x, y, z) = e^{Q(x,y,z)}$ is strictly convex.

(5) Check if the following equation defines z as a function $z = g(x, y)$ in a neighborhood of $(0, 0, 1)$. If it does, find $\frac{\partial g}{\partial x}$ and $\frac{\partial g}{\partial y}$ at $(0, 0, 1)$.

$$x^3 + y^3 + z^3 - xyz - 1 = 0.$$

(6) The same question at $(1, 0, 0)$ for the equation

$$e^z - z^2 - x^2 - y^2 = 0.$$

(7) Consider the system of equations

$$1 + (x + y)u - (2 + u)^{1+v} = 0$$
$$2u - (1 + xy)e^{u(x-1)} = 0.$$

Show that it defines u and v as functions of x and y near the point $(x, y, u, v) = (1, 1, 1, 0)$. Find $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}$ at this point.

(8) Write down a function on $\mathbf{R}^2$ with a critical point at $(0, 0)$ which is neither a local minimum nor a local maximum.

(9) Write down a function whose gradient at $(0, 0)$ is $(1, 3)$ and whose Hessian is $\begin{pmatrix} 2 & 1 \\ 1 & 8 \end{pmatrix}$.

(10) Consider the matrix

$$A = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 1 & 0 \\ 2 & 0 & -1 \end{pmatrix}.$$

Find an orthogonal matrix P such that $P^T AP$ is diagonal.

(11) State the spectral theorem.

(12) Let $f(x, y, z) = \sin(x + 2y)e^{z-y}$. Find the gradient and the Hessian of f. Write the second order Taylor approximation for f at $(0, 0, 0)$.

(13) Consider the function

$$f(x, y, z) = x^2 + y^2 + 3z^2 - xy + 2xz + yz.$$

Find all critical points and use the second derivative test to determine if each one is a local minimum, local maximum, or neither (or say that the test cannot determine the answer).

(14) Suppose a differentiable convex function f on $\mathbf{R}^n$ has a global maximum at a point p. Show that f must be a constant function.