ANALYSIS AND OPTIMIZATION: MIDTERM 2 PRACTICE PROBLEMS SOLUTIONS

SPRING 2016

Practice problem solutions

At times, I have only written the final answer or only sketched the solution. Let me know if something is unclear. I will add more explanation.

(1) Write the definition of a convex function. Let $f(\vec{x})$ and $g(\vec{x})$ be two convex functions on $\mathbf{R}^n$. Using the definition, show that the function $h(\vec{x})$ defined by

$$h(\vec{x}) = \max(f(\vec{x}), g(\vec{x}))$$

is also convex.

Solution. A function h is convex if for every $\vec{x}$ and $\vec{y}$ in the domain and λ in $[0,1]$, we have

$$\lambda h(\vec{x}) + (1-\lambda)h(\vec{y}) \geq h(\lambda\vec{x} + (1-\lambda)\vec{y}).$$

Let us show that $h = \max(f, g)$ is convex by verifying the above inequality. Since $h(\vec{x}) \geq f(\vec{x})$ and $h(\vec{y}) \geq f(\vec{y})$, we have

$$\lambda h(\vec{x}) + (1-\lambda)h(\vec{y}) \geq \lambda f(\vec{x}) + (1-\lambda)f(\vec{y}).$$

Since f is convex, we have

$$\lambda f(\vec{x}) + (1-\lambda)f(\vec{y}) \geq f(\lambda\vec{x} + (1-\lambda)\vec{y}).$$

Combining the two inequalities gives

$$\lambda h(\vec{x}) + (1-\lambda)h(\vec{y}) \geq f(\lambda\vec{x} + (1-\lambda)\vec{y}).$$

Similarly we get

$$\lambda h(\vec{x}) + (1-\lambda)h(\vec{y}) \geq g(\lambda\vec{x} + (1-\lambda)\vec{y}).$$

Combining the last two inequalities gives

$$\lambda h(\vec{x}) + (1-\lambda)h(\vec{y}) \geq \max(f(\lambda\vec{x} + (1-\lambda)\vec{y}), g(\lambda\vec{x} + (1-\lambda)\vec{y})),$$

which is the same as

$$\lambda h(\vec{x}) + (1-\lambda)h(\vec{y}) \geq h(\lambda\vec{x} + (1-\lambda)\vec{y}).$$

□

(2) Use Jensen's inequality to prove that for positive real numbers $x_1, \ldots, x_n$, we have

$$\sqrt[3]{\frac{x_1^3 + \cdots + x_n^3}{n}} \geq \frac{x_1 + \cdots + x_n}{n}.$$

Solution. Use Jensen's inequality for $f(x) = x^3$ (which is convex for $x > 0$) with all $\lambda_i = 1/n$. □

(3) Find the global minimum and maximum of the function $f(x,y) = 2x^3 + 4y^3$ on the set $S = \{x^2 + y^2 \leq 1\}$ by the following outline.

(a) Show that the maximum and the minimum exists.

Solution. f is a continuous function on a compact set S, so the max/min exist by the maximum theorem. □

(b) Using the gradient, find the possible points where the max/min could be achieved on the interior $\{x^2 + y^2 < 1\}$.

Solution. In the interior, the max/min can only be achieved when the gradient is equal to zero. The gradient is zero only at $(x,y) = (0,0)$. □

(c) Using Lagrange multipliers, find the possible points where the max/min could be achieved on the boundary $\{x^2 + y^2 = 1\}$.

Solution. The Lagrange multiplier problem is

$$\begin{aligned} 6x^2 &= 2\lambda x \\ 12y^2 &= 2\lambda y \\ x^2 + y^2 &= 1. \end{aligned}$$

The solutions are $(x,y) = (0,\pm 1), (\pm 1, 0), (2/\sqrt{5}, 1/\sqrt{5}), (-2/\sqrt{5}, -1/\sqrt{5})$. □

(d) Check all the possibilities.

Solution. The max is at $(0,1)$ and min at $(0,-1)$. □

(4) Let A be the matrix

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

(a) Write down the quadratic form $Q(x,y,z)$ associated with A.

Solution.

$$Q(x,y,z) = 2x^2 + 2y^2 + 2z^2 - 2xy - 2yz$$

□

(b) Show that the function $f(x,y,z) = e^{Q(x,y,z)}$ is strictly convex.

Solution. First, Q is strictly convex because the leading principal minors are positive. Now $e^{Q(x,y,z)}$ is the composition of a strictly convex function with a strictly increasing strictly convex function. □

(5) Check if the following equation defines z as a function $z = g(x,y)$ in a neighborhood of $(0,0,1)$. If it does, find $\frac{\partial g}{\partial x}$ and $\frac{\partial g}{\partial y}$ at $(0,0,1)$.

$$x^3 + y^3 + z^3 - xyz - 1 = 0.$$

(6) The same question at $(1,0,0)$ for the equation

$$e^z - z^2 - x^2 - y^2 = 0.$$

(7) Consider the system of equations

$$1+(x+y)u-(2+u)^{1+v}=0$$
$$2u-(1+xy)e^{u(x-1)}=0.$$

Show that it defines u and v as functions of x and y near the point $(x,y,u,v)=(1,1,1,0)$. Find $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}$ at this point.

Solution. The three problems above are applications of the implicit function theorem and the equation

$$\left(\frac{\partial F}{\partial Y}\right)\left(\frac{\partial Y}{\partial X}\right)=-\left(\frac{\partial F}{\partial X}\right),$$

where $F(X,Y)=0$ is the constraint equation and where the goal is to write Y as a function of X. By the implicit function theorem, this is possible if the $m\times m$ matrix $\left(\frac{\partial F}{\partial Y}\right)$ is invertible.

For example, denote the two equations in the last problem by f_1 and f_2. Then we get

$$\begin{aligned}\left(\frac{\partial(f_1,f_2)}{\partial(u,v)}\right)_{(1,1,1,0)} &= \begin{pmatrix} x+y-(1+v)(2+u)^v & -(2+u)^{1+v}\ln(2+u) \\ 2-(1+xy)(x-1)e^{u(x-1)} & 0\end{pmatrix}_{(1,1,1,0)} \\ &= \begin{pmatrix} 1 & -3\ln 3 \\ 2 & 0\end{pmatrix},\end{aligned}$$

which is invertible. Therefore, it is possible to write u and v as functions of x and y around $(1,1,1,0)$. To find the partials, solve

$$\begin{pmatrix} 1 & -3\ln 3 \\ 2 & 0\end{pmatrix}\begin{pmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y\end{pmatrix} = -\begin{pmatrix} u & u \\ -u(1+xy)e^{u(x-1)}-ye^{u(x-1)} & -xe^{u(x-1)}\end{pmatrix}_{(1,1,1,0)}$$

$$\begin{pmatrix} 1 & -3\ln 3 \\ 2 & 0\end{pmatrix}\begin{pmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y\end{pmatrix} = -\begin{pmatrix} 1 & 1 \\ -3 & -1\end{pmatrix}.$$

□

(8) Write down a function on $\mathbf{R}^2$ with a critical point at $(0,0)$ which is neither a local minimum nor a local maximum.

Solution. The easiest is to write down an indefinite quadratic form like x^2-y^2. □

(9) Write down a function whose gradient at $(0,0)$ is $(1,3)$ and whose Hessian is $\begin{pmatrix} 2 & 1 \\ 1 & 8\end{pmatrix}$.

Solution.

$$x+3y+\frac{1}{2}(2x^2+2xy+8y^2)$$

□

(10) Consider the matrix

$$A=\begin{pmatrix} 0 & 2 & 2 \\ 2 & 1 & 0 \\ 2 & 0 & -1\end{pmatrix}.$$

Find an orthogonal matrix P such that $P^T AP$ is diagonal.

Solution. The columns of P will be the *unit eigenvectors* of A. □

(11) State the spectral theorem.

Solution. For every symmetric matrix A, there exists an orthogonal matrix P such that $P^T AP$ is diagonal.

You may also state it using eigenvectors – every symmetric matrix has an orthogonal basis of eigenvectors. □

(12) Let $f(x, y, z) = \sin(x + 2y)e^{z-y}$. Find the gradient and the Hessian of f. Write the second order Taylor approximation for f at $(0, 0, 0)$.

(13) Consider the function

$$f(x, y, z) = x^2 + y^2 + 3z^2 - xy + 2xz + yz.$$

Find all critical points and use the second derivative test to determine if each one is a local minimum, local maximum, or neither (or say that the test cannot determine the answer).

(14) Suppose a differentiable convex function f on $\mathbf{R}^n$ has a global maximum at a point $\vec{p}$. Show that f must be a constant function.

Solution. Since f is convex, the graph of f lies above the tangent (hyper)plane at any point on the graph. But the tangent (hyper)plane at $\vec{p}$ is horizontal (since $\vec{p}$ is a maximum), and the graph of f cannot lie strictly above this hyperplane (since $\vec{p}$ is a maximum). So the graph must be this (hyper)plane. In other words, f is constant.

A less wordy and more math-y (and rigorous) way to write the above is as follows. Since f is convex, we have the inequality

$$f(\vec{x}) \geq f(\vec{p}) + \nabla f(p) \cdot (\vec{x} - \vec{p}).$$

Since $\vec{p}$ is a maximum, $\nabla f(p) = 0$, so we get

$$f(\vec{x}) \geq f(\vec{p}).$$

But since $\vec{p}$ is a maximum, we cannot have strict inequality, so we get

$$f(\vec{x}) = f(\vec{p})$$

for all $\vec{x}$. So f is constant. □