Calculus I: Practice Midterm II

April 1, 2014

Name: ____

- Write your solutions in the space provided. Continue on the back for more space.
- Show your work unless asked otherwise.
- Partial credit will be given for incomplete work.
- The exam contains 5 problems.
- **Good luck!**

1. The following two tables denote the values of two functions and their derivatives at various values of x.

x	-2	-1	0	1	2
$f(x)$	3	-2	1	4	5
$f'(x)$	-5	-3	2	3	7

x	-2	-1	0	1	2
$g(x)$	5	0	2	2	6
$g'(x)$	-6	0	1	1	2

(a) Let $a(x) = f(x)g(x)$. Find $a'(-1)$.

Solution: By the product rule,

$$a'(x) = f'(x)g(x) + f(x)g'(x).$$

So

$$a'(-1) = f'(-1)g(-1) + f(-1)g'(-1) = -3 \times 0 + -2 \times 0 = 0.$$

(b) Let $b(x) = f(g(x))$. Find $b'(0)$.

Solution: By the chain rule,

$$b'(x) = f'(g(x))g'(x).$$

So

$$b'(0) = f'(g(0))g'(0) = f'(2)g'(0) = 7.$$

(c) Let $c(x) = \frac{f(x)}{g(2x)}$. Find $c'(1)$.

Solution: By the quotient and chain rules,

$$\begin{aligned} c'(x) &= \frac{g(2x)f'(x) - 2g'(2x)f(x)}{g(2x)^2} \\ c'(1) &= \frac{g(2)f'(1) - 2g'(2)f(1)}{g(2)^2} \\ &= \frac{6 \times 3 - 2 \times 2 \times 4}{36} = \frac{1}{18} \end{aligned}$$

2. Compute the following.

(a) $f'(x)$ where $f(x) = \sin(2x) + \ln(x^2+1)$

Solution:

$$\begin{aligned} f'(x) &= \sin(2x)' + \ln(x^2+1)' \\ &= 2\cos(2x) + \frac{1}{x^2+1}\cdot 2x \quad \text{by the chain rule.} \\ &= 2\cos(2x) + \frac{2x}{x^2+1}. \end{aligned}$$

(b) $f'(1)$ where $f(x) = x^x + x^{-x}$.

Solution:

$$f'(x) = (x^x)' + (x^{-x})'.$$

For each term, we use logarithmic differentiation. For the first,

$$\begin{aligned} y &= x^x \\ \ln y &= x\ln x \\ (\ln y)' &= x\frac{1}{x} + \ln x \\ \frac{y'}{y} &= 1 + \ln x \\ y' &= y(1+\ln x) = x^x + x^x\ln x. \end{aligned}$$

Similarly,

$$\begin{aligned} z &= x^{-x} \\ \ln z &= -x\ln x \\ \frac{z'}{z} &= -(1+\ln x) \\ z' &= -x^{-x} - x^{-x}\ln x. \end{aligned}$$

Therefore,

$$f'(x) = x^x + x^x\ln x - x^{-x} - x^{-x}\ln x$$

So $f'(1) = 1 + 0 - 1 - 0 = 0$.

(c) $\lim_{x \to 0} \frac{\ln(x^2+1)}{\sin x}$

Solution: Since $\ln(x^2+1) \to \ln(1) = 0$ as $x \to 0$ and $\sin x \to 0$ as $x \to 0$, L'Hôpital's rule applies. We get

$$\begin{aligned}\lim_{x \to 0} \frac{\ln(x^2+1)}{\sin x} &= \lim_{x \to 0} \left(\frac{2x}{1+x^2} / \cos x \right) \\ &= \frac{0}{1}/1 = 0.\end{aligned}$$

(d) The slope of the tangent to the curve $x^3 + y^3 = xy + 1$ at the point $(1,1)$.

Solution: We use implicit differentiation. Treating y as an implicitly defined function of x and differentiating gives

$$3x^2 + 3y^2 \frac{dy}{dx} = y + x\frac{dy}{dx}.$$

Substituting $x = 1$ and $y = 1$ gives

$$\begin{aligned}3 + 3\frac{dy}{dx} &= 1 + \frac{dy}{dx} \\ 2\frac{dy}{dx} &= -2 \\ \frac{dy}{dx} &= -1.\end{aligned}$$

So the slope of the tangent is -1.

3. Maeby is inflating a spherical balloon. Suppose the radius of the balloon at a particular instant is 10 cm and its volume is increasing at the rate of 30 mL/s.

   (You may use Volume $= \frac{4}{3}\pi r^3$ and Surface area $= 4\pi r^2$)

   (a) At that instant, what is the rate at which the surface area is increasing?

Solution: Let V be the volume and S the surface area. We have $V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2$, where r is the radius. Differentiating with respect to t gives

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$
$$\frac{dS}{dt} = 8\pi r \frac{dr}{dt}.$$

Therefore, we see that

$$\frac{dS}{dt} = \frac{2}{r}\frac{dV}{dt}.$$

Substituting $r = 10$ cm and $\frac{dV}{dt} = 30\ \text{cm}^3/\text{s}$, we get

$$\frac{dS}{dt} = \frac{2}{10} \times 30 = 6\ \text{cm}^2/\text{s}.$$

   (b) Use your answer to approximate the surface area half a second afterwards.

Solution: At the given instant (say at $t = 0$), the surface area is

$$S(0) = 4\pi r^2 = 400\pi.$$

After one second, the surface area will be

$$S(1/2) \approx S(0) + \frac{1}{2} \times S'(0) = 400\pi + \frac{1}{2} \times 6 = (400\pi + 3)\ \text{cm}^2$$

4. Consider the function
$$f(x) = x^6 - 2x^3.$$

(a) Find the critical points of $f(x)$.

Solution: The critical points of $f(x)$ are x where $f(x)$ is not differentiable or where $f'(x) = 0$. Since $f(x)$ is differentiable for all x, the critical points are the solutions of

$$\begin{aligned} f'(x) &= 0 \\ 6x^5 - 6x^2 &= 0 \\ 6x^2(x^3 - 1) &= 0. \end{aligned}$$

So $x = 0$ and $x = 1$.

(b) Find the local minima and maxima of $f(x)$.

Solution: The local maxima and minima are among the critical points. To identify them, we look at the sign of $f'(x)$ for $x < 0$, $0 < x < 1$, and $1 < x$. We have
$$f'(x) = 6x^2(x^3 - 1).$$
For $x < 0$, this is negative, for $0 < x < 1$, this is negative, and for $1 < x$, this is positive. Hence $f(x)$ is decreasing in the interval $(-\infty, 0)$, decreasing in the interval $(0, 1)$ and increasing in the interval $(1, \infty)$.

We conclude that 0 is neither a local maximum nor a minimum and 1 is a local minimum.

(c) Find the global minima and maxima of $f(x)$ (if they exist).

Solution: Since $f(x)$ is decreasing in $(-\infty, 1)$, and increasing in $(1, \infty)$, it has a global minimum at $x = 1$.

Since $\lim_{x\to\infty} f(x) = +\infty$, $f(x)$ has no global maximum.

5. At a party, there are two loud-speakers of power 2 watt and 16 watt, separated by 10 meters. Suppose the volume of the sound emitted by a speaker is directly proportional to the power of the speaker and inversely proportional to the distance from the speaker. Where is the quietest spot on the line joining the two speakers?

Solution: To set up this problem, let us compute the loudness at distance x from the smaller speaker on the line joining the two speakers. The distance from the bigger speaker is then $10 - x$. The loudness is given by

$$L(x) = \frac{2}{x} + \frac{16}{(10-x)} = 2x^{-1} + 16(10-x)^{-1}.$$

We want to minimize $L(x)$ for $0 < x < 10$. Differentiation gives

$$L'(x) = -2x^{-2} + 16(10-x)^{-2}$$

Setting $L'(x) = 0$ gives

$$\begin{aligned} 2x^{-2} &= 16(10-x)^{-2} \\ \left(\frac{10-x}{x}\right)^2 &= 8 \\ 10 - x &= \sqrt{8}x \\ x &= \frac{10}{1+\sqrt{8}}. \end{aligned}$$

To ensure that this is the minimum, note that $L'(x)$ cannot change sign in $(0, 10/(1+\sqrt{8}))$. Therefore, $L(x)$ is either increasing or decreasing on $(0, 10/(1+\sqrt{8}))$. Since we have $\lim_{x\to 0} L(x) = +\infty$, we must have $L(x) > L(10/(1+\sqrt{8}))$ for x close to 0. Therefore, $L(x)$ is decreasing on $(0, 10/(1+\sqrt{8}))$.

Similarly, $L(x)$ is either increasing or decreasing on $(10/(1+\sqrt{8}), 10)$. Again, since we have $\lim_{x\to 10} L(x) = +\infty$, we must have $L(x) > L(10/(1+\sqrt{8}))$ for x close to 10. Therefore, $L(x)$ is increasing on $(10/(1+\sqrt{8}), 10)$.

Since $L(x)$ is decreasing on $(0, 10/(1+\sqrt{8}))$ and increasing on $(10/(1+\sqrt{8}), 10)$, the point $10/(1+\sqrt{8})$ is a global minimum of $L(x)$ on $(0, 10)$. In other words, the quietest spot corresponds to $x = 10/(1+\sqrt{8})$, that is, $10/(1+\sqrt{8})$ meters from the 2 watt speaker.