Sample exercises for the Final

December 20, 2009

1. Compute the following indefinite integrals:

   (a)

$$\int x\sin(3x^2+2)dx$$

Let $u = 3x^2 + 2$ then $du = 6x\,dx$ hence $\frac{du}{6} = 6\,dx$. Therefore

$$\begin{aligned}\int x\sin(3x^2+2)dx &= \int \sin u\frac{du}{6}\\ &= -\frac{1}{6}\cos u + C\\ &= -\frac{1}{6}\cos(3x^2+2) + C\end{aligned}$$

   (b)

$$\int \frac{x+3}{x^2}dx$$

$$\begin{aligned}\int \frac{x+3}{x^2}dx &= \int \frac{x}{x^2}dx + \int \frac{3}{x^2}dx =\\ &= ln|x| - 3\frac{1}{x} + C\end{aligned}$$

   (c)

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx$$

Let $u = e^{\sqrt{x}}$, then

$$du = e^{\sqrt{x}} \frac{1}{2\sqrt{x}} dx$$

therefore

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = \int 2\, du = 2u + C = 2e^{\sqrt{x}} + C$$

(d)

$$\int \frac{1+2x}{\sqrt{1-x^2}}$$

First of all let's divide the integral in two parts:

$$\int \frac{1+2x}{\sqrt{1-x^2}} = \int \frac{1}{\sqrt{1-x^2}} + \int \frac{2x}{\sqrt{1-x^2}}$$

we know how to deal with the first part, the antiderivative is arcsin. For the second part, le'ts make the substitution $u = 1-x^2$. Then $du = -2x\, dx$ therefore $-du = 2x\, dx$. Hence (for the second part):

$$\begin{aligned} \int \frac{2x}{\sqrt{1-x^2}} &= \int \frac{-1}{\sqrt{u}}\, du \\ &= \int -u^{-\frac{1}{2}}\, du \\ &= -2u^{\frac{1}{2}} + C \\ &= -2\sqrt{1-x^2} + C \end{aligned}$$

Hence the integral we began is

$$\int \frac{1+2x}{\sqrt{1-x^2}} = \arcsin x - 2\sqrt{1-x^2} + C$$

2. Compute the following integrals:

(a)

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{x \cos x}{1+x^4} dx$$

This function is odd and the integral is between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Hence the answer is 0.

(b)

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sqrt{\cos x}} dx$$

Let's make the substitution $u = \cos x$. Then $du = -\sin x \, dx$ and $u(0) = \cos 0 = 1$, $u(\frac{\pi}{4}) = \cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$

$$\begin{aligned}
\int_0^{\frac{\pi}{4}} \frac{\sin x}{\sqrt{\cos x}} dx &= -\int_1^{\frac{\sqrt{2}}{2}} \frac{1}{\sqrt{u}} \\
&= -\int_1^{\frac{\sqrt{2}}{2}} u^{-\frac{1}{2}} \\
&= -[2\sqrt{u}]_1^{\frac{1}{\sqrt{2}}} = 2 - \frac{2}{\sqrt[4]{2}}
\end{aligned}$$

(c)

$$\int_0^3 |x^2 - 4| \, dx$$

Let's start by noticing that $x^2 - 4 = (x-2)(x+2)$. We are interested in the interval $[0, 3]$. In this interval $x + 2$ is always > 0 but $x - 2 > 0$ for $x \in (2, 3]$, and $x - 2 < 0$ for $x \in [0, 2)$. Hence we have to split the integral into two parts!

$$\begin{aligned}
\int_0^3 |x^2 - 4| \, dx &= \int_0^2 |x^2 - 4| \, dx + \int_2^3 |x^2 - 4| \, dx \\
&= \int_0^2 -(x^2 - 4) \, dx + \int_2^3 (x^2 - 4) \, dx \\
&= \left[-\left(\frac{x^3}{3} - 4x \right) \right]_0^2 + \left[\left(\frac{x^3}{3} - 4x \right) \right]_2^3 \\
&= -\frac{8}{3} - 8 + \left((9 - 12 - \left(\frac{8}{3} - 8 \right) \right) \\
&= 13 - \frac{16}{3}
\end{aligned}$$

(d)

$$\int_{-1}^{\frac{1}{2}} \frac{x^2}{\sqrt{1-x}} dx$$

Let $u = 1 - x$. Then $du = -dx$. We still have to deal with the x^2 on the numerator: since $u = 1 - x$ get $x = 1 - u$ and $x^2 = 1 - 2u + u^2$.
We still have to compute the endpoints!! So $u(-1) = 1 - (-1) = 2$ and $u(\frac{1}{2}) = 1 - \frac{1}{2} = \frac{1}{2}$. Don't get worried by the fact that we are computing an integral where the upper limit of integration is smaller than the lower one (that's just minus the integral the other way around, so I'll just flip it). By substituting all of the information we have we get

$$\begin{aligned}
\int_{-1}^{\frac{1}{2}} \frac{x^2}{\sqrt{1-x}}\,dx &= \int_{2}^{\frac{1}{2}} -\frac{1-2u+u^2}{\sqrt{u}}\,du \\
&= \int_{\frac{1}{2}}^{2} (1-2u+u^2)u^{-\frac{1}{2}}\,du \\
&= \int_{\frac{1}{2}}^{2} (u^{-\frac{1}{2}} - 2u^{\frac{1}{2}} + u^{\frac{3}{2}}) \\
&= \left[2u^{\frac{1}{2}} - 2\frac{2}{3}u^{\frac{3}{2}} + \frac{5}{2}u^{\frac{5}{2}}\right]_{\frac{1}{2}}^{2} \\
&= 2\sqrt{2} - \frac{4}{3}\sqrt{8} + \frac{2}{5}\sqrt{32} - (\frac{2}{\sqrt{2}} - \frac{4}{3}\frac{1}{\sqrt{8}} + \frac{2}{5}\frac{1}{\sqrt{32}}) \\
&= 2\sqrt{2} - \frac{8}{3}\sqrt{2} + \frac{2}{5}4\sqrt{2} - (\sqrt{2} - \frac{\sqrt{2}}{3} + +\frac{1}{10\sqrt{2}})
\end{aligned}$$

(e)

$$\int_0^1 (y+3)^{100}\,dy$$

Let $u = y + 3$ then $du = dy$ so

$$\int_0^1 (y+3)^{100}\,dy = \int_3^4 u^{100}\,du$$

$$\left[\frac{u^{101}}{101}\right]_3^4 = \frac{4^{101}}{101} - \frac{3^{101}}{101}$$

3. State the fundamental theorem of calculus.
   Use it to compute

$$\frac{d}{dx}\int_x^{3x-1} \tan(2t-1)\sqrt{t}dt$$

   Is this computation correct:

$$\int_{-1}^{2} \frac{1}{x^2}dx = \left[\frac{-1}{x}\right]_{-1}^{2} = -\frac{1}{2} - 1 = -\frac{3}{2}$$

Solution: Let f be a continuous function on $[a, b]$. Let

$$g(x) = \int_a^x f(t)dt \qquad\qquad a \leq x \leq b$$

Then g is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$. Moreover, if F is any antiderivative of f then

$$\int_a^b f(t)\, dt = F(b) - F(a)$$

To compute the derivative we first have to put the integral in better shape.

$$\begin{aligned}\int_x^{3x-1} \tan(2t-1)\sqrt{t}\, dt &= \int_x^0 \tan(2t-1)\sqrt{t}\, dt + \int_0^{3x-1} \tan(2t-1)\sqrt{t}\, dt \\ &= -\int_0^x \tan(2t-1)\sqrt{t}\, dt + \int_0^{3x-1} \tan(2t-1)\sqrt{t}\, dt\end{aligned}$$

Now we have to take the derivative. Remember to apply the chain rule for the second summand! (if you set $g(x) = \int_0^x \tan(2t-1)\sqrt{t}dt$ then the second summand is $g(3x-1)$!)

$$\begin{aligned}\frac{d}{dx}\int_x^{3x-1} \tan(2t-1)\sqrt{t}\, dt &= -\tan(2x-1)\sqrt{x} + 3\tan(2(3x-1)-1)\sqrt{3x-1} \\ &= -\tan(2x-1)\sqrt{x} + 3\tan(6x-3)\sqrt{3x-1}\end{aligned}$$

As for what is wrong with the computation: we cannot apply the fundamental theorem of Calculus in this case since $\frac{1}{x^2}$ is not continuous on $[-1, 2]$.

4. If f is continuous and $\int_1^{22} f(x)dx = 3$, compute

$$\int_0^7 f(3x+1)dx$$

Solution: This is a nice way to see if you can do substitution integrals. Let's compute $\int_0^7 f(3x+1)dx$ by making the substitution $u = 3x+1$ then $du = 3dx$ hence $\frac{du}{3} = dx$. Moreover, $u(0) = 1$ and $u(7) = 22$. Hence

$$\int_0^7 f(3x+1)dx = \int_1^{22} f(u)\frac{du}{3} = \frac{1}{3}\int_1^{22} f(u)\, du = \frac{1}{3} \cdot 3 = 1$$

5. Find the volume of the solid obtained by considering the region bounded by $y = x^3$ and $x = 1$ and $y = 0$ and and rotating it along the line $y = -2$.
Solution: We will integrate with respect to x. The cross section at height x is an annulus of inner radius 2 and the outer radius is $2 + x^3$. Hence get

$$\begin{aligned} V(S) &= \int_0^1 \pi((2 + x^3)^2 - 2^2)\, dx \\ &= \int_0^1 \pi(4x^3 + x^6)\, dx \\ &= \pi \left[x^4 + \frac{x^7}{7} \right]_0^1 = \pi \left(1 + \frac{1}{7} \right) \end{aligned}$$

6. Find the points on the hyperbola $y^2 - x^2 = 4$ closest to the point $(2, 0)$
Solution: First of all, the distance of a point (x, y) from the point $(2, 0)$ is

$$d = \sqrt{(x - 2)^2 + y^2}$$

and if the point is on the hyperbola we know that $y^2 = x^2 + 4$ so we get a function of x

$$d(x) = \sqrt{(x - 2)^2 + x^2 + 4}$$

Optional (but might simplify computations): minimizing the distance is the same thing is minimizing the distance squared (since x^2 is an increasing function!). Hence we just have to minimize

$$f(x) = d(x)^2 = (x - 2)^2 + x^2 + 4 = 2x^2 - 4x + 8$$

The domain for x is $(-\infty, +\infty)$. By taking the derivative we get

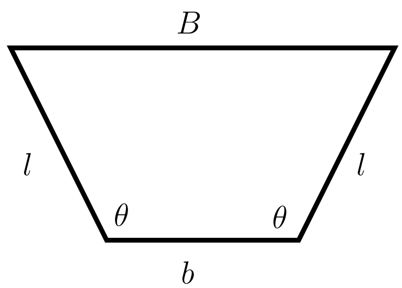
$$f'(x) = 4x - 4$$

which is zero only at $x = 1$. The second derivative here is $f'(1) = 4$ so f has a min here.
If $x = 1$ then $y^2 = x^2 + 4 = 5$ hence $y = \pm\sqrt{5}$ hence there are two such points, $(1, \sqrt{5})$ and $(1, -\sqrt{5})$.

7. Find the volume of the solid obtained by rotating about the line $x = -1$ the region between $y = \frac{1}{x}$ and $x = 1$ and $x = 3$ and $y = 0$.
Solution: This is most easily done by cylindrical shells:

$$V = \int_0^3 2\pi(2 + x)\frac{1}{x}\, dx = 2\pi \int_1^3 \left(\frac{2}{x} + 1 \right) dx = 2\pi \left[2 \ln x + x \right]_1^3 = 2\pi(2 \ln 3 + 2)$$

8. Consider the following trapezoid:



(b and l are fixed numbers, B and θ are not). Find the angle θ that maximizes the area (this problem is hard!!).
Solution:
Let h be the height of the trapezoid. Let A be the area of the trapezoid. Then we know

$$A = \frac{(b+B)h}{2}$$

We know that b is a fixed number so we just have to express B and h is terms of the constants b and l and the one variable θ.
First of all

$$h = l \cdot \cos(\theta - \frac{\pi}{2})$$

just to make things look better I *can* notice that $\cos(\theta - \frac{\pi}{2}) = \cos(\theta)\cos(-\frac{\pi}{2}) - \sin(\theta)\sin(-\frac{\pi}{2}) = \sin(\theta)$ hence $h = l \cdot \sin\theta$ but I don't even need to do this! (it just gets slightly more complicated otherwise). Also

$$B = b + 2l \cdot \sin(\theta - \frac{\pi}{2}) = b - 2l\cos\theta$$

Hence we find

$$A(\theta) = \frac{(2b - 2l \cdot \cos\theta) l \cdot \sin\theta}{2} = (b - l\cos\theta) l \sin\theta = bl\sin\theta - l^2\cos\theta\sin\theta$$

The domain for θ is $\theta \in [0, \pi]$. This is a close interval nad we know how to find the max: we just check critical points and the endpoints of the interval.
For the endpoints: $A(0) = 0$ and $A(\pi) = 0$.
Let's differentiate $A(\theta)$:

$$A'(\theta) = bl\cos\theta - l^2(-\sin^2\theta + \cos^2\theta) = bl\cos\theta + l^2\sin^2\theta - l^2\cos^2\theta$$

Wow, this is hard!! Let's substitute $\sin^2\theta = 1 - \cos^2\theta$ so that we get a quadratic equation in $\cos\theta$!

$$A'(\theta) = bl\cos\theta + l^2 - 2l^2\cos^2\theta = l(b\cos\theta + l - 2l\cos^2\theta)$$

hence $A'(\theta) = 0$ when

$$\cos\theta = \frac{b \pm \sqrt{b^2 + 8l^2}}{4l}$$

OK, so we know there has to be at least one critical point (Rolle's theorem!). But are there two of them?

Well, notice that, since $b, l > 0$, $b^2 + 8l^2 = b^2 + 4l^2 + 4l^2 > 4l^2$ hence $\sqrt{b^2 + 8l^2} > \sqrt{4l^2} = 2l$. Hence

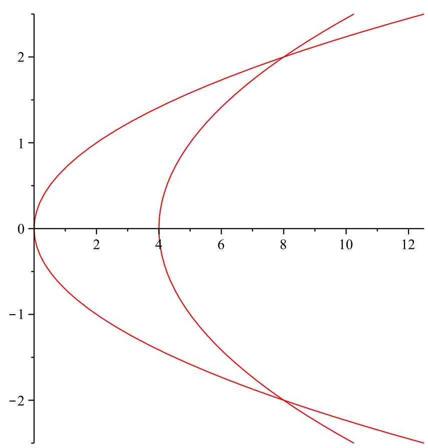
$$\frac{b + \sqrt{b^2 + 4l^2}}{2l} > \frac{2l}{2l} = 1$$

so there can't be a number θ such that $\cos\theta = \frac{b+\sqrt{b^2+4l^2}}{2l}$!!!

Hence the critical point **must** be $\theta = \arccos\frac{b-\sqrt{b^2+4l^2}}{2l}$. And this is definitely the max since the area of the trapezoid is bigger than zero at least for some θ so there must be a local max. (or just think of it like that: the area of this theta must really be bigger than zero-proof by picture!).

OK, this was way too hard. But it was fun, wasn't it? In the final I would at least give you specific numbers to deal with. so don't worry if you couldn't do this by yourself.

9. Find the area enclosed between the two curves $x = 2y^2$ and $x = 4 + y^2$.



Solution: We have to integrate with respect to y. But first let's find the points in which the two parabolas meet: $2y^2 = 4 + y^2 \Rightarrow y^2 = 4$ hence $y = 2$ and $y = -2$. Therefore

$$A = \int_{-2}^{2} (4 + y^2 - 2y^2)\,dy = \int_{-2}^{2} (4 - y^2)\,dy = \left[4y - \frac{y^3}{3}\right]_{-2}^{2} = 16 - \frac{16}{3}$$