2. First we draw a line passing through Dubbo and Sydney. We approximate the directional derivative at Dubbo in the direction of Sydney by the average rate of change of temperature between the points where the line intersects the contour lines closest to Dubbo. In the direction of Sydney, the temperature changes from $30°$C to $27°$C. We estimate the distance between these two points to be approximately 120 km, so the rate of change of maximum temperature in the direction given is approximately $\frac{27-30}{120} = -0.025°\text{C/km}$.

4. $f(x,y) = x^3y^4 + x^4y^3 \quad \Rightarrow \quad f_x(x,y) = 3x^2y^4 + 4x^3y^3$ and $f_y(x,y) = 4x^3y^3 + 3x^4y^2$. If $\mathbf{u}$ is a unit vector in the direction of $\theta = \frac{\pi}{6}$, then from Equation 6, $D_{\mathbf{u}}f(1,1) = f_x(1,1)\cos\left(\frac{\pi}{6}\right) + f_y(1,1)\sin\left(\frac{\pi}{6}\right) = 7 \cdot \frac{\sqrt{3}}{2} + 7 \cdot \frac{1}{2} = \frac{7+7\sqrt{3}}{2}$.

7. $f(x,y) = \sin(2x+3y)$

(a) $\nabla f(x,y) = \dfrac{\partial f}{\partial x}\,\mathbf{i} + \dfrac{\partial f}{\partial y}\,\mathbf{j} = [\cos(2x+3y)\cdot 2]\,\mathbf{i} + [\cos(2x+3y)\cdot 3]\,\mathbf{j} = 2\cos(2x+3y)\,\mathbf{i} + 3\cos(2x+3y)\,\mathbf{j}$

(b) $\nabla f(-6,4) = (2\cos 0)\,\mathbf{i} + (3\cos 0)\,\mathbf{j} = 2\,\mathbf{i} + 3\,\mathbf{j}$

(c) By Equation 9, $D_{\mathbf{u}}\,f(-6,4) = \nabla f(-6,4)\cdot\mathbf{u} = (2\,\mathbf{i} + 3\,\mathbf{j}) \cdot \frac{1}{2}\left(\sqrt{3}\,\mathbf{i} - \mathbf{j}\right) = \frac{1}{2}\left(2\sqrt{3} - 3\right) = \sqrt{3} - \frac{3}{2}$.

20. $f(x,y,z) = xy + yz + zx \quad \Rightarrow \quad \nabla f(x,y,z) = \langle y+z, x+z, y+x\rangle$, so $\nabla f(1,-1,3) = \langle 2,4,0\rangle$. The unit vector in the direction of $\overrightarrow{PQ} = \langle 1,5,2\rangle$ is $\mathbf{u} = \frac{1}{\sqrt{30}}\langle 1,5,2\rangle$, so $D_{\mathbf{u}}\,f(1,-1,3) = \nabla f(1,-1,3)\cdot\mathbf{u} = \langle 2,4,0\rangle \cdot \frac{1}{\sqrt{30}}\langle 1,5,2\rangle = \frac{22}{\sqrt{30}}$.

22. $f(s,t) = te^{st} \quad \Rightarrow \quad \nabla f(s,t) = \left\langle te^{st}(t), te^{st}(s) + e^{st}(1)\right\rangle = \left\langle t^2e^{st}, (st+1)e^{st}\right\rangle$.

$\nabla f(0,2) = \langle 4,1\rangle$ is the direction of maximum rate of change, and the maximum rate is $|\nabla f(0,2)| = \sqrt{16+1} = \sqrt{17}$.

25. $f(x,y,z) = \sqrt{x^2+y^2+z^2} \quad \Rightarrow$

$$\nabla f(x,y,z) = \left\langle \tfrac{1}{2}(x^2+y^2+z^2)^{-1/2}\cdot 2x, \tfrac{1}{2}(x^2+y^2+z^2)^{-1/2}\cdot 2y, \tfrac{1}{2}(x^2+y^2+z^2)^{-1/2}\cdot 2z\right\rangle$$

$$= \left\langle \frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}}\right\rangle,$$

$\nabla f(3,6,-2) = \left\langle \frac{3}{\sqrt{49}}, \frac{6}{\sqrt{49}}, \frac{-2}{\sqrt{49}}\right\rangle = \left\langle \frac{3}{7}, \frac{6}{7}, -\frac{2}{7}\right\rangle$. Thus the maximum rate of change is

$|\nabla f(3,6,-2)| = \sqrt{\left(\frac{3}{7}\right)^2 + \left(\frac{6}{7}\right)^2 + \left(-\frac{2}{7}\right)^2} = \sqrt{\frac{9+36+4}{49}} = 1$ in the direction $\left\langle \frac{3}{7}, \frac{6}{7}, -\frac{2}{7}\right\rangle$ or equivalently $\langle 3,6,-2\rangle$.

33. $\nabla V(x,y,z) = \langle 10x - 3y + yz, xz - 3x, xy \rangle$, $\nabla V(3,4,5) = \langle 38, 6, 12 \rangle$

(a) $D_{\mathbf{u}} V(3,4,5) = \langle 38, 6, 12 \rangle \cdot \frac{1}{\sqrt{3}} \langle 1, 1, -1 \rangle = \frac{32}{\sqrt{3}}$

(b) $\nabla V(3,4,5) = \langle 38, 6, 12 \rangle$, or equivalently, $\langle 19, 3, 6 \rangle$.

(c) $|\nabla V(3,4,5)| = \sqrt{38^2 + 6^2 + 12^2} = \sqrt{1624} = 2\sqrt{406}$

44. Let $F(x,y,z) = xy + yz + zx$. Then $xy + yz + zx = 5$ is a level surface of F and $\nabla F(x,y,z) = \langle y+z, x+z, x+y \rangle$.

(a) $\nabla F(1,2,1) = \langle 3, 2, 3 \rangle$ is a normal vector for the tangent plane at $(1,2,1)$, so an equation of the tangent plane is $3(x-1) + 2(y-2) + 3(z-1) = 0$ or $3x + 2y + 3z = 10$.

(b) The normal line has direction $\langle 3, 2, 3 \rangle$, so parametric equations are $x = 1 + 3t$, $y = 2 + 2t$, $z = 1 + 3t$, and symmetric equations are $\dfrac{x-1}{2} = \dfrac{y-2}{1} = \dfrac{z-1}{3}$.

55.

The hyperboloid $x^2 - y^2 - z^2 = 1$ is a level surface of $F(x,y,z) = x^2 - y^2 - z^2$ and $\nabla F(x,y,z) = \langle 2x, -2y, -2z \rangle$ is a normal vector to the surface and hence a normal vector for the tangent plane at (x,y,z). The tangent plane is parallel to the plane $z = x + y$ or $x + y - z = 0$ if and only if the corresponding normal vectors are parallel, so we need a point (x_0, y_0, z_0) on the hyperboloid where $\langle 2x_0, -2y_0, -2z_0 \rangle = c\,\langle 1, 1, -1 \rangle$ or equivalently $\langle x_0, -y_0, -z_0 \rangle = k\,\langle 1, 1, -1 \rangle$ for some $k \neq 0$. Then we must have $x_0 = k$, $y_0 = -k$, $z_0 = k$ and substituting into the equation of the hyperboloid gives $k^2 - (-k)^2 - k^2 = 1 \quad \Leftrightarrow \quad -k^2 = 1$, an impossibility. Thus there is no such point on the hyperboloid.

2. (a) $D = g_{xx}(0,2)\, g_{yy}(0,2) - [g_{xy}(0,2)]^2 = (-1)(1) - (6)^2 = -37$. Since $D < 0$, g has a saddle point at $(0,2)$ by the Second Derivatives Test.

(b) $D = g_{xx}(0,2)\, g_{yy}(0,2) - [g_{xy}(0,2)]^2 = (-1)(-8) - (2)^2 = 4$. Since $D > 0$ and $g_{xx}(0,2) < 0$, g has a local maximum at $(0,2)$ by the Second Derivatives Test.

(c) $D = g_{xx}(0,2)\, g_{yy}(0,2) - [g_{xy}(0,2)]^2 = (4)(9) - (6)^2 = 0$. In this case the Second Derivatives Test gives no information about g at the point $(0,2)$.

3. In the figure, a point at approximately $(1, 1)$ is enclosed by level curves which are oval in shape and indicate that as we move away from the point in any direction the values of f are increasing. Hence we would expect a local minimum at or near $(1, 1)$. The level curves near $(0, 0)$ resemble hyperbolas, and as we move away from the origin, the values of f increase in some directions and decrease in others, so we would expect to find a saddle point there.

To verify our predictions, we have $f(x, y) = 4 + x^3 + y^3 - 3xy \quad \Rightarrow \quad f_x(x, y) = 3x^2 - 3y$, $f_y(x, y) = 3y^2 - 3x$. We have critical points where these partial derivatives are equal to 0: $3x^2 - 3y = 0$, $3y^2 - 3x = 0$. Substituting $y = x^2$ from the first equation into the second equation gives $3(x^2)^2 - 3x = 0 \quad \Rightarrow \quad 3x(x^3 - 1) = 0 \quad \Rightarrow \quad x = 0$ or $x = 1$. Then we have two critical points, $(0, 0)$ and $(1, 1)$. The second partial derivatives are $f_{xx}(x, y) = 6x$, $f_{xy}(x, y) = -3$, and $f_{yy}(x, y) = 6y$, so $D(x, y) = f_{xx}(x, y)\, f_{yy}(x, y) - [f_{xy}(x, y)]^2 = (6x)(6y) - (-3)^2 = 36xy - 9$. Then $D(0, 0) = 36(0)(0) - 9 = -9$, and $D(1, 1) = 36(1)(1) - 9 = 27$. Since $D(0, 0) < 0$, f has a saddle point at $(0, 0)$ by the Second Derivatives Test. Since $D(1, 1) > 0$ and $f_{xx}(1, 1) > 0$, f has a local minimum at $(1, 1)$.

44. Let x, y, z, be the positive numbers. Then $x + y + z = 12$ and we want to minimize $x^2 + y^2 + z^2 = x^2 + y^2 + (12 - x - y)^2 = f(x, y)$ for $0 < x, y < 12$. $f_x = 2x + 2(12 - x - y)(-1) = 4x + 2y - 24$, $f_y = 2y + 2(12 - x - y)(-1) = 2x + 4y - 24$, $f_{xx} = 4$, $f_{xy} = 2$, $f_{yy} = 4$. Then $f_x = 0$ implies $4x + 2y = 24$ or $y = 12 - 2x$ and substituting into $f_y = 0$ gives $2x + 4(12 - 2x) = 24 \quad \Rightarrow \quad 6x = 24 \quad \Rightarrow \quad x = 4$ and then $y = 4$, so the only critical point is $(4, 4)$. $D(4, 4) = 16 - 4 > 0$ and $f_{xx}(4, 4) = 4 > 0$, so $f(4, 4)$ is a local minimum. $f(4, 4)$ is also the absolute minimum [compare to the values of f as $x, y \to 0$ or 12] so the numbers are $x = y = z = 4$.

51. Let the dimensions be x, y and z, then minimize $xy + 2(xz + yz)$ if $xyz = 32{,}000 \text{ cm}^3$. Then $f(x, y) = xy + [64{,}000(x + y)/xy] = xy + 64{,}000(x^{-1} + y^{-1})$, $f_x = y - 64{,}000x^{-2}$, $f_y = x - 64{,}000y^{-2}$. And $f_x = 0$ implies $y = 64{,}000/x^2$; substituting into $f_y = 0$ implies $x^3 = 64{,}000$ or $x = 40$ and then $y = 40$. Now $D(x, y) = [(2)(64{,}000)]^2 x^{-3} y^{-3} - 1 > 0$ for $(40, 40)$ and $f_{xx}(40, 40) > 0$ so this is indeed a minimum. Thus the dimensions of the box are $x = y = 40$ cm, $z = 20$ cm.