Calculus III: Midterm I

February 21, 2013

Name: Anand Deopurkar (Solutions)

Circle one: Section 6 (11:40–12:55) Section 7 (2:40–3:55)

- Write your answers in the space provided. Use the backside if you need more space.
- You must **show your work** unless explicitly asked otherwise.
- Partial credit will be given for incomplete solutions.
- The exam contains 5 problems.
- **Good luck!**

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

1. Write true or false. *No justification is needed.*

(a) (2 points) If $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$ then we must have $\vec{b} = \vec{c}$.

True False

$\bar{a} \times \bar{b} = \bar{a} \times \bar{c} \Rightarrow \quad \bar{a} \times (\bar{b} - \bar{c}) = 0.$

Then $\bar{b} - \bar{c}$ is in the same/opposite direction as $\bar{a}$.

It need not be 0.

(b) (2 points) The two planes defined by $x + y + z = 0$ and $x - 2y + z = 0$ are perpendicular.

True False

The normal directions are

$\langle 1,1,1 \rangle$ and $\langle 1,-2,1 \rangle$ whose dot

product is zero.

(c) (2 points) The surface described by $x^2 + 2y^2 = 1 + z$ is a hyperbolic paraboloid.

True False

Putting $z = k$ (constant), we see that

the traces are ellipses.

(d) (2 points) $e^{2+i\pi}$ is a real number.

True False

Argument of $e^{2+i\pi} = e^2 . e^{i\pi}$ is π,

which corresponds to the (negative) real

direction.

(e) (2 points) If two lines in $\mathbb{R}^3$ do not intersect, they must be parallel.

True False

They can be skew.

2. Find the angle between

(a) (3 points) $\mathbf{i}+\mathbf{k}$ and $\mathbf{j}+\mathbf{k}$.

$$\cos\theta = \frac{(i+k)\cdot(j+k)}{|i+k|\,|j+k|} = \frac{1}{\sqrt{2}\cdot\sqrt{2}} = \frac{1}{2}. \quad \text{So } \theta = \frac{\pi}{3}$$

(b) (3 points) $\langle 1,1,0\rangle \times \langle 0,1,1\rangle$ and $\langle 1,0,-1\rangle$.

$$\langle 1,1,0\rangle \times \langle 0,1,1\rangle = \begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= i - j + k = \langle 1,-1,1\rangle$$

$\langle 1,-1,1\rangle \cdot \langle 1,0,-1\rangle = 0$

So the angle is $\frac{\pi}{2}$.

(c) (4 points) The lines through the origin defined by $x = \frac{y}{2} = z$ and $x = y, z = 0$.

Converting to parametric form:

$$x = \frac{y}{2} = z = t \Rightarrow \left.\begin{aligned} x &= t \\ y &= 2t \\ z &= t \end{aligned}\right\} \text{ Direction vector is } \langle 1,2,1\rangle.$$

$$\begin{aligned} x = y &= t \\ z &= 0 \end{aligned} \Rightarrow \left.\begin{aligned} x &= t \\ y &= t \\ z &= 0 \end{aligned}\right\} \text{ Direction vector is } \langle 1,1,0\rangle.$$

θ = Angle between the direction vectors

$$\cos\theta = \frac{\langle 1,2,1\rangle \cdot \langle 1,1,0\rangle}{|\langle 1,2,1\rangle|\,|\langle 1,1,0\rangle|} = \frac{3}{\sqrt{6}\cdot\sqrt{2}} = \frac{\sqrt{3}}{2}$$

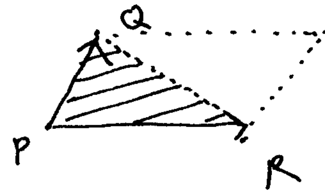
So $\theta = \frac{\pi}{6}$

3. Consider the following three points in $\mathbb{R}^3$

$$P = (1,1,1), \quad Q = (1,2,1) \quad R = (2,1,2).$$

(a) (5 points) Find the area of the triangle PQR.

Area of $\Delta PQR = \frac{1}{2}$ Area of parallelogram $\overline{PQ}, \overline{PR}$



$= \frac{1}{2} \, |\overline{PQ} \times \overline{PR}|.$

$\overline{PQ} = (1,2,1) - (1,1,1) = (0,1,0) = j$

$\overline{PR} = (2,1,2) - (1,1,1) = (1,0,1) = i + k$

$\overline{PQ} \times \overline{PR} = j \times (i + k) = j \times i + j \times k = -k + i = i - k$

So Area of $\Delta PQR = \frac{1}{2} \, |i - k| = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

(b) (5 points) Does the plane passing through P, Q and R also pass through the origin? Show your work.

We can take $\overline{PQ} \times \overline{PR}$ as the normal vector for this plane. From the previous part,

$\overline{n} = \overline{PQ} \times \overline{PR} = i - k = \langle 1, 0, -1 \rangle.$

So the plane through PQR = plane through $(1,1,1)$ normal to $\langle 1, 0, -1 \rangle$.

i.e. $\big((x,y,z) - (1,1,1)\big) \cdot \langle 1, 0, -1 \rangle = 0$

$(x-1) - (z-1) = 0$

$x - z = 0$

Since the origin satisfies this equation, it lies on the plane. In other words, the plane through P, Q, R also passes through the origin.

4. (a) (4 points) Find all the complex numbers satisfying

$$x^2 - x + 1 = 0.$$

Using the quadratic formula,

$$x = \frac{1 \pm \sqrt{1-4}}{2}$$

$$= \frac{1 \pm \sqrt{-3}}{2} = \frac{1}{2} \pm \frac{\sqrt{3}\sqrt{-1}}{2}$$

$$= \frac{1}{2} \pm \frac{\sqrt{3}}{2} i.$$

So the two solutions are $\frac{1}{2} + \frac{\sqrt{3}}{2} i$ and $\frac{1}{2} - \frac{\sqrt{3}}{2} i$

(b) (6 points) Pick one x that you found in the previous part and calculate x^{99}.

Let us pick $x = \frac{1}{2} + \frac{\sqrt{3}}{2} i$.

For computing x^{99}, it is better to express x in polar form.

Now $|x| = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$.

$\arg x = \theta$ for which $\cos\theta = \frac{1}{2}$, $\sin\theta = \frac{-\sqrt{3}}{2}$

$= \frac{\pi}{3}$.

That is, $x = e^{i\pi/3} = \cos\frac{\pi}{3} + i \sin\frac{\pi}{3}$.

Then $x^{99} = e^{\frac{i\pi}{3} \times 99}$

$$= e^{33\pi i} = \cos(33\pi) + i \sin(33\pi)$$

$$= \cos(\pi) + i \sin(\pi)$$

$$= -1$$

5. (a) (3 points) Let P be the plane passing through the point $(1, 1, 1)$ and perpendicular to the vector $\mathbf{i} + \mathbf{j}$. Write an equation for P.

For (x,y,z) to be on this plane,

$$((x,y,z) - (1,1,1)) \cdot \langle i+j \rangle = 0$$

$$\Rightarrow (x-1, y-1, z-1) \cdot \langle 1,1,0 \rangle = 0$$

$$\Rightarrow (x-1) + (y-1) = 0$$

$$\Rightarrow x+y = 2.$$

(b) (3 points) Write parametric equations for the line L joining $(1, 2, 3)$ and $(3, 2, 1)$.

Direction vector $v = (3,2,1) - (1,2,3) = \langle 2, 0, -2 \rangle$.

So the line is given by $\langle 1,2,3 \rangle + t \langle 2,0,-2 \rangle$

or

$$x = 1 + 2t$$
$$y = 2$$
$$z = 3 - 2t$$

(c) (4 points) Use your equations from the previous parts to find the point of intersection of L and P.

The point of intersection satisfies the equations for L and the equation for P.

Substituting the parametric expressions for x, y, z in the equation for L, we get

$$x+y = 2 \Rightarrow (1+2t) + 2 = 2 \Rightarrow 2t = -1$$

$$\Rightarrow t = \frac{-1}{2}.$$

In that case, the point $(x,y,z) = (0,2,4)$.

So the point of intersection is $(0,2,4)$.