Calculus III: Midterm I

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Circle one: Section 6 (11:40 12:55) Section 7 (2:40-3:55)

- Calculators or other computing devices are not allowed.
- Write your answers in the space provided. Use the backside if you need more space.
- You must show your work unless explicitly asked otherwise.
- Partial credit will be given for incomplete solutions.
- The exam contains 5 problems.
- **Good luck!**

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

1. Write true or false. *No justification is needed.*

(a) (2 points) If two lines in $\mathbb{R}^3$ do not intersect, they must be parallel.

True False

They can be skew.

(b) (2 points) The surface described by $x^2 + 2y^2 = z$ is a hyperbolic paraboloid.

True False

The traces when $z = k$ (constant) are ellipses.

(c) (2 points) The two planes defined by $x + 2y + z = 0$ and $x + y + z = 0$ are perpendicular.

True False

The normal vectors are $\langle 1, 2, 1\rangle$ and $\langle 1, 1, 1\rangle$, whose dot product is not zero.

(d) (2 points) If $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$ then we must have $\vec{b} = \vec{c}$.

True False

$\bar{a} \cdot \bar{b} = \bar{a} \cdot \bar{c}$ means

$\bar{a} \cdot (\bar{b} - \bar{c}) = 0$, that is

$\bar{a}$ and $\bar{b} - \bar{c}$ are perpendicular. $\bar{b} - \bar{c}$ need not be zero.

(e) (2 points) $e^{3-4\pi i}$ is a real number.

True False

Argument of $e^{3-4\pi i}$ is 4π, which corresponds to the (positive) real direction.

2. Find the angle between

(a) (3 points) $\langle 1, 0, 1\rangle \times \langle 0, 1, 1\rangle$ and $\langle 1, -1, 0\rangle$.

$$\langle 1,0,1\rangle \times \langle 0,1,1\rangle = \begin{vmatrix} i & j & k \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -i-j+k = \langle -1,-1,1\rangle$$

$$\langle -1,-1,1\rangle \cdot \langle 1,-1,0\rangle = 0$$

So the angle is $\frac{\pi}{2}$.

(b) (3 points) $\mathbf{i} + \mathbf{j}$ and $\mathbf{j} + \mathbf{k}$.

$$\cos\theta = \frac{(i+j)\cdot(j+k)}{|i+j|\,|j+k|} = \frac{1}{\sqrt{2}\cdot\sqrt{2}} = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$$

(c) (4 points) The lines through the origin defined by $\frac{x}{2} = y = z$ and $x = y, z = 0$.

Writing the lines in parametric form:

$$\frac{x}{2} = y = z = t \Rightarrow \left.\begin{aligned} x &= 2t \\ y &= t \\ z &= t \end{aligned}\right\} \text{Direction vector} = \langle 2,1,1\rangle$$

$$\begin{aligned} x=y&=t \\ z&=0 \end{aligned} \Rightarrow \left.\begin{aligned} x &= t \\ y &= t \\ z &= 0 \end{aligned}\right\} \text{Direction vector} = \langle 1,1,0\rangle$$

So the angle is given by

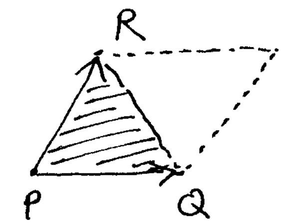
$$\cos\theta = \frac{\langle 2,1,1\rangle \cdot \langle 1,1,0\rangle}{|\langle 2,1,1\rangle|\,|\langle 1,1,0\rangle|} = \frac{3}{\sqrt{6}\sqrt{2}} = \frac{\sqrt{3}}{2}$$

that is $\theta = \frac{\pi}{6}$

3. Consider the following three points in $\mathbb{R}^3$

$$P = (1, 1, 1), \quad Q = (1, 3, 1) \quad R = (2, 1, 0).$$

(a) (5 points) Find the area of the triangle PQR.



Area of ΔPQR =

$\frac{1}{2}$ Area of parallelogram spanned by $\overline{PQ}$ and $\overline{PR}$

$= \frac{1}{2} |\overline{PQ} \times \overline{PR}|$

$\overline{PQ} = (1,3,1) - (1,1,1) = (0,2,0) = 2j$

$\overline{PR} = (2,1,0) - (1,1,1) = (1,0,-1) = i - k$

$\overline{PQ} \times \overline{PR} = 2j \times (i-k) = 2\, j\times i - 2 j \times k = -2k - 2i$

$|\overline{PQ} \times \overline{PR}| = \sqrt{4+4} = \sqrt{8}$ so area of $\Delta PQR = \frac{1}{2}\sqrt{8} = \sqrt{2}$.

(b) (5 points) Does the plane passing through P, Q and R also pass through the origin? Show your work.

We can take $\overline{PQ} \times \overline{PR}$ as the normal vector to this plane. From our previous work,

$\overline{PQ} \times \overline{PR} = -2i - 2k = \langle -2, 0, -2 \rangle$.

So the plane through P, Q, R = Plane through $(1,1,1)$ perpendicular to $\langle -2, 0, -2 \rangle$.

For (x,y,z) to be on this plane,

$(\langle x,y,z \rangle - \langle 1,1,1 \rangle) \cdot (-2, 0, -2) = 0$

$\Rightarrow \langle (x-1), (y-1), (z-1) \rangle \cdot \langle -2, 0, -2 \rangle = 0$

$\Rightarrow 2(x-1) + 2(z-1) = 0$ that is

$x + z = 2$. The origin does not satisfy this equation. So the plane does not pass through the origin.

4. (a) (4 points) Find all the complex numbers satisfying

$$x^2 - x + 1 = 0.$$

Using the quadratic formula,

$$x = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm \sqrt{-3}}{2}$$

$$= \frac{1}{2} \pm \frac{\sqrt{3}\sqrt{-1}}{2}$$

$$= \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

So the two solutions are

$$\frac{1}{2} + \frac{\sqrt{3}}{2} i \quad \text{and} \quad \frac{1}{2} - \frac{\sqrt{3}}{2} i.$$

(b) (6 points) Pick one x that you found in the previous part and calculate x^{66}.

We better write x in polar form.

Let us take $x = \frac{1}{2} + \frac{\sqrt{3}}{2} i$.

Then $|x| = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$

$\arg x = \theta$ is such that $\cos\theta = \frac{1}{2}$, $\sin\theta = \frac{\sqrt{3}}{2}$

$= \frac{\pi}{3}$.

That is $x = e^{i\pi/3} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$

So $x^{66} = e^{\frac{i\pi}{3} \times 66} = e^{22\pi i}$

$$= \cos(22\pi) + i\sin(22\pi)$$

$$= \cos(0) + i\sin(0)$$

$$= 1$$

5. (a) (3 points) Write parametric equations for the line L joining $(0,1,2)$ and $(2,1,0)$.

Direction vector $v = (2,1,0) - (0,1,2)$
$= (2,0,-2)$.

Line : $(0,1,2) + t\,(2,0,-2)$ or

$$\left.\begin{aligned} x &= 2t \\ y &= 1 \\ z &= 2-2t. \end{aligned}\right\} L.$$

(b) (3 points) Let P be the plane passing through the point $(1,1,1)$ and perpendicular to the vector $\mathbf{i} + \cancel{\mathbf{k}}$ $\mathbf{j}$. Write an equation for P.

For $\langle x,y,z\rangle$ to be on the plane,

$((x,y,z) - (1,1,1)) \cdot (i+j) = 0$

$\Rightarrow$ $(x-1, y-1, z-1) \cdot (1,1,0) = 0$

$\Rightarrow$ $x-1+y-1 = 0$

$\Rightarrow$ $\boxed{x+y=2}$

(c) (4 points) Use your equations from the previous parts to find the point of intersection of L and P.

The point of intersection must satisfy the equations for L and the eq^n for P. Substituting x,y,z from L :

$2t + 1 = 2 \Rightarrow t = \frac{1}{2}$

$\Rightarrow (x,y,z) = (1,1,1)$.

So the point of intersection is $\underline{\underline{(1,1,1)}}$.