Calculus III: Midterm II

Name: ____

- Read the problems carefully.
- You must show your work unless asked otherwise.
- Partial credit will be given for incomplete work.
- The exam contains 5 problems.
- The last page is the formula sheet, which you may detach.
- **Good luck!**

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

1. (10 points) Write true or false. No justification is needed.

(a) The curve parametrized by $\langle t^3, t^3+1, 2t^3\rangle$ is a straight line.

Solution: True. As t goes from $-\infty$ to ∞, so does t^3, so the curve is the same as $\langle t, t+1, 2t\rangle$.

True False

(b) The acceleration vector is always contained in the osculating plane.

Solution: True. The acceleration is a combination of T and N, which both lie in the osculating plane.

True False

(c) The graph of $f(x, y) = xy + 2$ is a plane.

Solution: False. The graph is the surface defined by $z = xy + 2$, which is not a linear equation.

True False

(d) If the speed is constant, then the acceleration must be zero.

Solution: False. For example, the speed is constant in a uniform circular motion, but the acceleration is certainly not zero.

True False

(e) The domain of the function $f(x, y) = \ln(x^2 + y^2)$ is the entire $\langle x, y\rangle$ plane.

Solution: False. The origin $(0, 0)$ is not in the domain.

True False

2. Consider the curve C parametrized by

$$r(t) = \sin^2 t\ \mathbf{i} + t\ \mathbf{j} + \cos^2 t\ \mathbf{k}.$$

(a) (4 points) Write parametric equations for the tangent line to C at $t = 0$.

Solution: The tangent line at $t = 0$ is the line through $r(0)$ in the direction $r'(0)$. We have

$$r'(t) = 2\sin t \cos t\mathbf{i} + \mathbf{j} - 2\cos t \sin t\mathbf{k},$$

so that

$$r(0) = \langle 0, 0, 1\rangle$$
$$r'(0) = \langle 0, 1, 0\rangle.$$

Hence the tangent line is

$$x = 0 \quad y = t \quad z = 1.$$

(b) (4 points) Write an equation for the normal plane to C at $t = 0$.

Solution: The normal plane at $t = 0$ is the plane through $r(0)$ perpendicular to $r'(0)$. From our calculation of $r'(0)$, we get that the equation is

$$(\langle x, y, z\rangle - \langle 0, 0, 1\rangle) \cdot \langle 0, 1, 0\rangle = 0$$
$$\text{that is: } y = 0.$$

(c) (2 points) The curve C lies on a plane. Find the equation of this plane.

Solution: To find a plane containing C, we must find a linear relation in x, y, and z that is always satisfied when $x = \sin^2 t$, $y = t$ and $z = \cos^2 t$. Such a relation is

$$x + z = 1.$$

Hence C lies on the plane defined by $x + z = 1$.

3. When two asteroids collide, their fates depend on their speed at the time of impact and also the angle of collision. Suppose the asteroids have trajectories $\overrightarrow{a}(t) = \langle t, 2t - t^2, t\rangle$ and $\overrightarrow{b}(t) = \langle t, t, -\cos(\pi t)\rangle$.

   (a) (7 points) Find the speeds of A and B at the time of collision.

   Solution: First, we must find the time of collision. This is a value of t for which $\overrightarrow{a}(t) = \overrightarrow{b}(t)$, that is

   $$\langle t, 2t - t^2, t\rangle = \langle t, t, -\cos(\pi t).$$

   Comparing the first coordinate gives nothing. The second coordinate gives $2t - t^2 = t$ or $t^2 - t = 0$. Therefore $t = 0$ or $t = 1$. However, the third coordinates match only for $t = 1$. Hence the time of collision is $t = 1$.
   Next, we have

   $$\overrightarrow{a}'(t) = \langle 1, 2 - 2t^2, 1\rangle \implies \overrightarrow{a}'(1) = \langle 1, 0, 1\rangle$$
   $$\overrightarrow{b}'(t) = \langle 1, 1, \pi\sin(\pi t) \implies \overrightarrow{b}'(1) = \langle 1, 1, 0\rangle.$$

   Hence the speed of A at the time of collision is $|\overrightarrow{a}'(1)| = \sqrt{2}$ and the speef of B at the time of collision is $|\overrightarrow{b}'(1)| = \sqrt{2}$.

   (b) (3 points) Find the angle of collision.

   Solution: The angle of collision is just the angle between the trajectories at $t = 1$, which is the same as the angle between the respective tangent vectors at $t = 1$. The angle θ between $\overrightarrow{a}'(1) = \langle 1, 0, 1\rangle$ and $b'(1) = \langle 1, 1, 0\rangle$ can be calculated by

   $$\cos(\theta) = \frac{\overrightarrow{a}'(1) \cdot \overrightarrow{b}'(1)}{|\overrightarrow{a}'(1)||\overrightarrow{b}'(1)|} = \frac{1}{2}.$$

   Hence $\theta = \pi/3$.

4. Let C be the intersection of the cylinder $x^2 + y^2 = 4$ and the plane $y + z = 0$.

(a) (3 points) Find a vector function $\overrightarrow{r}(t)$ that parametrizes C.

Solution: Since $z = -y$ and x and y by themselves describe a circle of radius 2, we can take

$$\overrightarrow{r}(t) = \langle 2\cos t, 2\sin t, -2\sin t\rangle.$$

(b) (7 points) Find point(s) on C at which the curvature is the maximum. What is the maximum value of the curvature?

Solution: Firstly, we have to compute the curvature (which will depend on t) and then find the value of t that maximizes it. We know

$$\begin{aligned}
\kappa &= \frac{r' \times r''}{|r'|^3} \\
r'(t) &= \langle -2\sin t, 2\cos t, -2\cos t\rangle \\
r''(t) &= \langle -2\cos t, -2\sin t, 2\sin t\rangle \\
r'(t) \times r''(t) &= \langle 0, 4, -4\rangle \\
|r'(t)| &= \sqrt{4\sin^2 t + 4\cos^2 t + 4\cos^2 t} \\
&= 2\sqrt{1 + \cos^2 t} \\
\text{so that } \kappa &= \frac{|\langle 0, 4, -4\rangle|}{8(1 + \cos^2 t)^{3/2}} = \frac{1}{\sqrt{2}(1 + \cos^2 t)^{3/2}}.
\end{aligned}$$

Since the numerator is constant, κ is maximum when the denominator is minimum, or equivalently, when $1 + \cos^2 t$ is minimum. But the lowest value of $\cos^2 t$ is 0, achieved when $t = \pi/2, 3\pi/2, \ldots$. Hence the maximum value of κ is

$$\kappa_{\max} = \frac{1}{\sqrt{2}}.$$

It is achieved when $t = \pi/2, 3\pi/2, \ldots$, which correspond to the points

$$P_1 = \langle 0, 2, -2\rangle \text{ and } P_2 = \langle 0, -2, 2\rangle$$

on C.

5. This question concerns the helical road described by the equations

$$x = 3\sin t \quad y = 3\cos t \quad z = 4t.$$

The units for distance in this problem are meters and the units for time are seconds.

(a) (4 points) A car going up along this road starts at $(0, 3, 0)$ and travels 10π meters. Where is it now?

Solution: The starting point corresponds to $t = 0$. Suppose the car is at the point corresponding to $t = T$ after going 10π meters. Then the length of the road from $t = 0$ to $t = T$ is 10π. In other words,

$$10\pi = \int_0^T |\langle 3\cos t, -3\sin t, 4\rangle| \; dt = \int_0^T 5 \; dt = 5T.$$

Therefore $T = 2\pi$. Hence the car is at

$$\langle 0, 3, 8\pi\rangle.$$

(b) (6 points) The road is banked (tilted) to handle acceleration of up to 4 m/s^2 in the normal direction. What should be the speed limit on this piece of road? Justify your answer.

Solution: Suppose we have a car going at the speed s. The constraint is that the normal component of its acceleration must be at most 4. But we know that $a_N = \kappa s^2$, where κ is the curvature of the road. Therefore, we must have $\kappa s^2 \leq 4$, that is $s \leq \sqrt{\frac{4}{\kappa}}$ So the speed limit should be $\sqrt{\frac{4}{\kappa}}$.
Setting $r(t) = \langle 3\sin t, 3\cos t, 4t\rangle$, we have

$$\begin{aligned}
\kappa &= \frac{|r' \times r''|}{|r'|^3} \\
r'(t) &= \langle 3\cos t, -3\sin t, 4\rangle \\
r''(t) &= \langle -3\sin t, -3\cos t, 0\rangle \\
r'(t) \times r''(t) &= \langle 12\cos t, 12\sin t, -9\rangle \\
\kappa &= \frac{\sqrt{12^2 + 9^2}}{\sqrt{3^2 + 4^2}^3} = \frac{3}{25}.
\end{aligned}$$

Hence the speed limit should be

$$\sqrt{\frac{4}{\kappa}} = \sqrt{\frac{100}{3}} = \frac{10}{\sqrt{3}}.$$

LIST OF USEFUL IDENTITIES

1. Derivatives

(1) $\frac{d}{dx}x^n = nx^{n-1}$

(2) $\frac{d}{dx}\sin x = \cos x$

(3) $\frac{d}{dx}\cos x = -\sin x$

(4) $\frac{d}{dx}\tan x = \sec^2 x$

(5) $\frac{d}{dx}\cot x = -\csc^2 x$

(6) $\frac{d}{dx}\sec x = \sec x \tan x$

(7) $\frac{d}{dx}\csc x = -\csc x \cot x$

(8) $\frac{d}{dx}e^x = e^x$

(9) $\frac{d}{dx}\ln|x| = \frac{1}{x}$

(10) $\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$

(11) $\frac{d}{dx}\arccos x = \frac{-1}{\sqrt{1-x^2}}$

(12) $\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$

2. Trigonometry

(1) $\sin^2 x + \cos^2 x = 1$

(2) $\tan^2 x + 1 = \sec^2 x$

(3) $1 + \cot^2 x = \csc^2 x$

(4) $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$

(5) $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

(6) $\sin^2 x = \frac{1-\cos 2x}{2}$

(7) $\cos^2 x = \frac{1+\cos 2x}{2}$.

3. Space curves

For a parametric space curve given by $\overline{r}(t)$

(1) Curvature $\quad \kappa = \dfrac{|r'(t) \times r''(t)|}{|r'(t)|^3}$.

(2) Tangent component of acceleration $\quad a_T = |r'(t)|' = \dfrac{r'(t) \cdot r''(t)}{|r'(t)|}$.

(3) Normal component of acceleration $\quad a_N = \kappa|r'(t)|^2 = \dfrac{|r'(t) \times r''(t)|}{|r'(t)|}$.