Calculus III: Practice Final

Name: ____________________

Circle one: Section 6 Section 7 .

- Read the problems carefully.
- Show your work unless asked otherwise.
- Partial credit will be given for incomplete work.
- The exam contains 10 problems.
- The last page is the formula sheet, which you may detach.
- **Good luck!**

Question:	1	2	3	4	5	6	7	8	9	10	Total
Points:	10	10	10	10	10	10	10	10	10	10	100
Score:											

1. (10 points) Circle True or False. No justifation is needed.

(a) The curve traced by $\langle \cos^2(t), \sin^2(t) \rangle$ is a circle.

True False

Solution: False.

(b) The plane $3x + 2y - z = 0$ is perpendicular to the line $x = 3t, y = 2t, z = -t$.

True False

Solution: True.

(c) The function

$$f(x, y, z) = \begin{cases} \frac{\sin(x+y+z)}{x+y+z} & \text{if } x + y + z \neq 0 \\ 1 & \text{if } x + y + z = 0 \end{cases}$$

is continuous at $(0, 0, 0)$.

True False

Solution: True.

(d) If the acceleration is constant, then the trajectory must be a straight line.

True False

Solution: False.

(e) The complex number e^{2+3i} has magnitude 2.

True False

Solution: False.

2. In the following, compute $V \cdot W$, $V \times W$, and the cosine of the angle between V and W.

(a) (5 points) $V = \langle 2, -1, 1 \rangle, \quad W = \langle 1, 3, -2 \rangle$.

Solution:

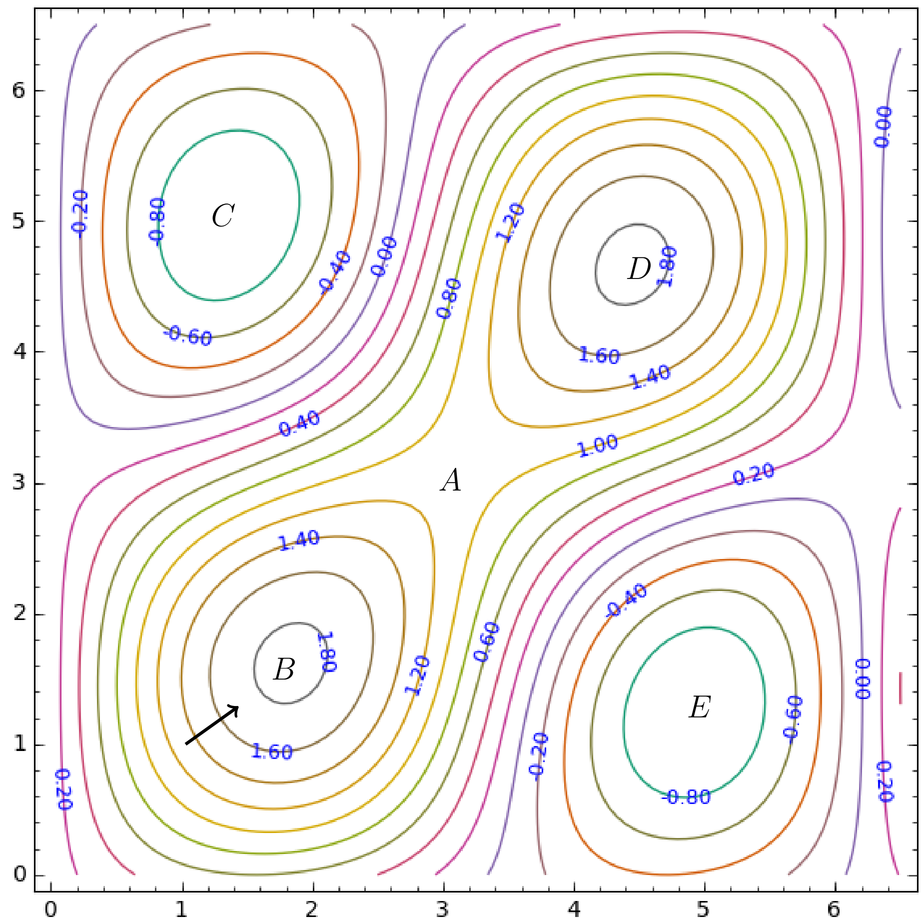
$$\begin{aligned} V \cdot W &= 2 - 3 + 2 = -3 \\ V \times w &= \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ 1 & 3 & -2 \end{bmatrix} \\ &= -\mathbf{i} + 5\mathbf{j} - 7\mathbf{k} = \langle -1, 5, -7 \rangle. \\ \cos\theta &= \frac{V \cdot W}{|V||W|} \\ &= \frac{-3}{\sqrt{6} \cdot \sqrt{14}} \\ &= \frac{-\sqrt{3}}{2\sqrt{7}}. \end{aligned}$$

(b) (5 points) $V = \mathbf{i} + 3\mathbf{j}, \quad W = 3\mathbf{j} - 2\mathbf{k}$.

Solution:

$$\begin{aligned} V \cdot W &= 9 \\ V \times W &= \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 0 \\ 0 & 3 & -2 \end{bmatrix} \\ &= -6\mathbf{i} + 2\mathbf{j} - 3\mathbf{k} \\ \cos\theta &= \frac{V \cdot W}{|V||W|} \\ &= \frac{9}{\sqrt{10} \cdot \sqrt{13}} \\ &= \frac{9}{\sqrt{130}}. \end{aligned}$$

3. Use the contour plot of $f(x, y)$ to answer the questions. No justification is needed.



(a) (3 points) Mark any three critical points of f. Label them A, B, and C. Identify whether they are local minima, local maxima or saddle points.

Solution: A: Saddle; B, D: Local maxima, C, E: Local minima.

(b) (2 points) Draw a vector at $(1, 1)$ indicating the direction of ∇f at $(1, 1)$.

(c) (3 points) Determine the sign of

1. $\frac{\partial f}{\partial x}(3, 4)$: Positve.
2. $\frac{\partial f}{\partial y}(2, 3)$: Negative.
3. $D_u f(5, 3)$ where u is the South–East direction: Negative.

(d) (2 points) Give a (admittedly rough) numerical estimate of $\frac{\partial f}{\partial x}(1, 1)$.

Solution: $\partial f/\partial x(1, 1) \approx 0.2/0.2 = 1.0$

4. (a) (5 points) Write parametric equations for the tangent line at $\langle 1, 0, 1\rangle$ to the curve traced by $\langle t^2, \ln t, t^3\rangle$.

Solution: Let $r(t) = \langle t^2, \ln t, t^3\rangle$. Then $r(1) = \langle 1, 0, 1\rangle$.

$$\begin{aligned} r'(t) &= \langle 2t, 1/t, 3t^2\rangle \\ r'(1) &= \langle 2, 1, 3\rangle. \end{aligned}$$

The tangent line is the line through $\langle 1, 0, 1\rangle$ in the direction of $\langle 2, 1, 3\rangle$, that is:

$$(x, y, z) = \langle 1, 0, 1\rangle + t\langle 2, 1, 3\rangle.$$

In other words,

$$x = 1 + 2t, \quad y = t, \quad z = 1 + 3t.$$

(b) (5 points) Write an equation of the normal plane to the curve at the same point.

Solution: The normal plane is the plane through $\langle 1, 0, 1\rangle$ perpendicular to $r'(1) = \langle 2, 1, 3\rangle$. The equation is

$$\begin{aligned} &\langle x - 1, y, z - 1\rangle \cdot \langle 2, 1, 3\rangle = 0 \\ \implies &2(x - 1) + y + 3(z - 1) = 0 \\ \implies &2x + y + 3z = 5. \end{aligned}$$

5. (a) (5 points) Let $f(x,y) = xy/(x^2+y^2)$. Find $\lim_{(x,y)\to(0,0)} f(x,y)$ or show that the limit does not exist.

Solution: The limit does not exist.

Approach $(0,0)$ along the x axis $(y=0)$. Then $f(x,y) = f(x,0) = 0$. Hence the limit is 0. On the other hand, approach $(0,0)$ along the line $x = y$. Then $f(x,y) = f(x,x) = 1/2$. Hence the limit is $1/2$. Since f approaches two different values along two different paths to $(0,0)$, its limit at $(0,0)$ does not exist.

(b) (5 points) Let $f(x,y) = \int_0^{xy} e^{t^2}\,dt$. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

Solution: Let

$$G(T) = \int_0^T e^{t^2}\,dt.$$

Then $\frac{dG}{dT} = e^{T^2}$ by the Fundamental Theorem of Calculus. Also, $f = G(T)$ if $T = xy$. By the chain rule,

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial G}{\partial T}\cdot\frac{\partial T}{\partial x}\\ &= e^{T^2}\cdot y\\ &= ye^{x^2y^2}.\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial G}{\partial T}\cdot\frac{\partial T}{\partial y}\\ &= e^{T^2}\cdot x\\ &= xe^{x^2y^2}.\end{aligned}$$

6. A ball of unit mass is thrown with the initial velocity of $\mathbf{i}+\mathbf{j}$. It experiences the force of gravity of magnitude 10 units in the $-\mathbf{j}$ direction and a force due to the wind of magnitude 1 unit in the $\mathbf{i}$ direction. Suppose the ball is initially at $(0,4)$.

   (a) (7 points) Find the position of the ball at time t.

   Solution: We have

$$\begin{aligned}\text{Acceleration } \vec{a} &= \mathbf{i} - 10\mathbf{j}\\ \text{Therefore, velocity } \vec{v}(t) &= \int \vec{a}\; dt\\ &= t\mathbf{i} - 10t\mathbf{j} + \vec{c}.\end{aligned}$$

   Since $\vec{v}(0) = \mathbf{i}+\mathbf{j}$, we get $\vec{c} = \mathbf{i}+\mathbf{j}$. Therefore

$$\begin{aligned}\vec{v}(t) &= (1+t)\mathbf{i} + (1-10t)\mathbf{j}\\ \text{Therefore, position } \vec{r}(t) &= \int \vec{v}(t)\; dt\\ &= \left(t + \frac{t^2}{2}\right)\mathbf{i} + (t - 5t^2)\mathbf{j} + \vec{c}.\end{aligned}$$

   Since $\vec{r}(0) = 4\mathbf{j}$, we get $\vec{c} = 4\mathbf{j}$. Therefore,

$$\vec{r}(t) = \left(t + \frac{t^2}{2}\right)\mathbf{i} + (4 + t - 5t^2)\mathbf{j}.$$

   In other words, at time t, the ball is at $x = t + t^2/2$ and $y = 4 + t - 5t^2$.

   (b) (3 points) Where is the ball when it hits the ground?

   Solution: The ball hits the ground when $y = 0$. That is

$$\begin{aligned}&4 + t - 5t^2 = 0\\ &5t^2 - t - 4 = (5t+4)(t-1) = 0.\end{aligned}$$

   The only positive solution is $t = 1$. So the ball hits the ground at $t = 1$. At this point, it is at $x = 1 + 1/2 = 3/2$ and $y = 0$.

7. Suppose $u = e^{xy}$ where $x = st + s + t$ and $y = st - s - t$.

(a) (2 points) Find the value of u when $s = 2$ and $t = 2$.

Solution: Then $x = 8$ and $y = 0$, so that $e^{xy} = e^0 = 1$.

(b) (8 points) Find an approximate numerical value of u when $s = 2.01$ and $t = 1.98$.

Solution: We know that

$$\Delta u \approx \frac{\partial u}{\partial s}\Delta s + \frac{\partial u}{\partial t}\Delta t.$$

By the chain rule,

$$\begin{aligned}\frac{\partial u}{\partial s} &= \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial s} \\ &= ye^{xy}(t+1) + xe^{xy}(t-1) \\ &= 8 \text{ at } s = 2, t = 2.\end{aligned}$$

Likewise,

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial t} \\ &= ye^{xy}(s+1) + xe^{xy}(s-1) \\ &= 8 \text{ at } s = 2, t = 2.\end{aligned}$$

Hence

$$\begin{aligned}\Delta u &\approx 8 \cdot 0.01 + 8 \cdot (-0.02) \\ &= -0.08.\end{aligned}$$

In other words, $u \approx 1 - 0.08 = 0.92$.
(By the way, the exact answer is $u = 0.92192448707\ldots$.)

8. (10 points) Find all the critical points of the function $f(x, y) = x^3 + y^3 - 3xy$. Determine if they are local maxima, local minima or saddle points.

Solution: We know that (x, y) is a critical point precisely when $\nabla f(x, y) = 0$. Therefore,

$$\frac{\partial f}{\partial x} = 3x^2 - 3y = 0 \implies y = x^2$$
$$\frac{\partial f}{\partial y} = 3y^2 - 3x = 0 \implies x = y^2.$$

Substituting $x = y^2$ in the first equation, we get $y = y^4$, that is $y = 0$ or $y = 1$. If $y = 0$, then we get $x = 0$ from the second equation, and if $y = 1$, then we get $x = 1$ from the second equation. Hence the critical points are

$$(0, 0) \text{ and } (1, 1).$$

To determine their type, we compute the matrix of second derivatives

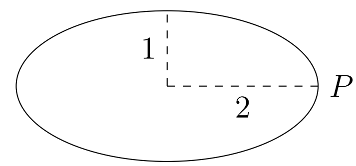
$$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix} = \begin{bmatrix} 6x & -3 \\ -3 & 6y \end{bmatrix}.$$

At $(0, 0)$, we get $\begin{bmatrix} 0 & -3 \\ -3 & 0 \end{bmatrix}$, whose determinant is $D = -9$. Since $D < 0$, this is a saddle point.

At $(1, 1)$, we get $\begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix}$, whose determinant is $D = 27$. Since $D > 0$, and $f_{xx} > 0$, this is a local minimum.

9. (10 points) A race track is in the shape of an ellipse with minor radius 1 km and major radius 2 km as shown. A car is going along this track at a constant speed of 100 km/h. Find the tangent and normal component of its accelaration when it as at P.

Race track



Solution: Let $r(t)$ be the position of the car at time t. We know that at time t, the tangent and normal components of accelaration are given by

$$a_T = |r'(t)|' \text{ and}$$
$$a_N = \kappa |r'(t)|^2,$$

where κ is the curvature of the trajectory at the point $r(t)$. Since the speed $|r'(t)|$ is constant, we immediately get that

$$a_T = 0.$$

What remains is

$$a_N = \kappa |r'(t)|^2 = 100^2 \kappa.$$

To get the answer, we must compute the curvature of the ellipse at the point P.

To compute the curvature, we first conveniently parametrize the ellipse. For example,

$$x = 2\cos t, \quad y = \sin t.$$

Then

$$\begin{aligned}\kappa &= \frac{|\langle -2\sin t, \cos t\rangle \times \langle -2\cos t, -\sin t\rangle|}{|\langle -2\sin t, \cos t\rangle|^3} \\ &= \frac{|(2\sin^2 t + 2\cos^2 t)\mathbf{k}|}{(4\sin^2 t + \cos^2 t)^{3/2}} \\ &= \frac{2}{(4\sin^2 t + \cos^2 t)^{3/2}} \\ &= 2 \text{ at } P \text{ (set } t = 0).\end{aligned}$$

Therefore,

$$a_N = 2 \cdot 100^2 = 20000 \text{ km/h}^2.$$

10. (10 points) You want to design a cylindrical cup that can hold 100π ml coffee. To minimize the material to be used, you decide to minimize the surface area. What is the radius and height of the optimal cup? (Ignore the thickness of the walls.)

Solution: Let the radius of the cup be r and the height h. We want

$$\text{Volume} = \pi r^2 h = 100\pi, \text{ that is } r^2 h = 100.$$

We want to minimize

$$\begin{aligned}\text{Surface area} &= \pi r^2(\text{bottom}) + 2\pi r h(\text{side}) \\ &= \pi(r^2 + 2rh).\end{aligned}$$

So, we want to minimize $f(r,h) = r^2 + 2rh$ subject to $g(r,h) = r^2 h = 100$. By the method of Lagrange multipliers, the extremal values are attained when

$$\nabla f = \lambda \nabla g.$$

We have

$$\nabla f = \langle 2r + 2h, 2r \rangle, \quad \nabla g = \langle 2rh, r^2 \rangle.$$

These two are multiples of each other when

$$\frac{2r + 2h}{2rh} = \frac{2r}{r^2}.$$

Simplifying and cross-multiplying gives

$$r + h = 2h \implies r = h.$$

So the only critical point is when $r = h$. To find this value, we use the constraint

$$r^2 h = 100.$$

This gives $r = h = 100^{1/3}$.

(By the way, $r = h$ would result in a very bizarre coffee cup!)

LIST OF USEFUL IDENTITIES

1. Derivatives

(1) $\frac{d}{dx}x^n = nx^{n-1}$

(2) $\frac{d}{dx}\sin x = \cos x$

(3) $\frac{d}{dx}\cos x = -\sin x$

(4) $\frac{d}{dx}\tan x = \sec^2 x$

(5) $\frac{d}{dx}\cot x = -\csc^2 x$

(6) $\frac{d}{dx}\sec x = \sec x \tan x$

(7) $\frac{d}{dx}\csc x = -\csc x \cot x$

(8) $\frac{d}{dx}e^x = e^x$

(9) $\frac{d}{dx}\ln|x| = \frac{1}{x}$

(10) $\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$

(11) $\frac{d}{dx}\arccos x = \frac{-1}{\sqrt{1-x^2}}$

(12) $\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$

2. Trigonometry

(1) $\sin^2 x + \cos^2 x = 1$

(2) $\tan^2 x + 1 = \sec^2 x$

(3) $1 + \cot^2 x = \csc^2 x$

(4) $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$

(5) $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

(6) $\sin^2 x = \frac{1-\cos 2x}{2}$

(7) $\cos^2 x = \frac{1+\cos 2x}{2}$.

3. Space curves

For a parametric space curve given by $\overline{r}(t)$

(1) Curvature $\quad \kappa = \dfrac{|r'(t) \times r''(t)|}{|r'(t)|^3}$.

(2) Tangent component of acceleration $\quad a_T = |r'(t)|' = \dfrac{r'(t) \cdot r''(t)}{|r'(t)|}$.

(3) Normal component of acceleration $\quad a_N = \kappa |r'(t)|^2 = \dfrac{|r'(t) \times r''(t)|}{|r'(t)|}$.