Calculus III: Practice problems

May 6, 2013

The problems are mostly taken from [1], sometimes with minor modifications. Some miscelleneous problems are from [2].

1 Geometry of vectors

1. In the following, compute $V+W$ and $2V-3W$ and $|V|^2$ and $V \cdot W$ and $V \times W$ and $\cos\theta$, where θ is the angle between V and W.
   (a) $V=(1,1,1), W=(-1,-1,-1)$.
   (b) $V=\mathbf{i}+\mathbf{j}, W=\mathbf{j}-\mathbf{k}$.
   (c) $V=\mathbf{i}-2\mathbf{j}+\mathbf{k}, W=\mathbf{i}+\mathbf{j}-2\mathbf{k}$.

2. True or false in three dimensions:
   (a) If both U and V make a 30 degree angle with W, so does $U+V$.
   (b) If they make a 90 degree angle with W, so does $U+V$.
   (c) If they make a 90 degree angle with W they are perpendicular to each other: $U \cdot V = 0$.

3. From $W=(1,2,3)$ subtract a multiple of $V=(1,1,1)$ so that $W-cV$ is perpendicular to V. (Find c).

4. (a) What is the sum V of the twelve vectors from the center of a clock to the hours?
   (b) If the 4 o'clock vector is removed, find the sum V for the other eleven vectors.
   (c) If the vectors to 1, 2, 3 are cut in half, find the sum V for the twelve vectors.

5. Find parametric equations for the line through $P=(1,2,4)$ and $Q=(5,5,4)$. Probably your speed is 5; change the equations so the speed is 10.

6. Find an equation for any plane perpendicular to the line in the problem above. Also find (parametric) equations for any line that is perpendicular to this line.

7. Find the angle between the diagonal of cube and (a) an edge (b) the diagonal of a face (c) another diagonal of the cube.

8. Find an equation of the plane:
   (a) through $(0,0,0)$ perpendicular to $i+j-k$,
   (b) through $(0,0,0)$ and $(1,0,0)$ and $(0,1,1)$.
   (c) through $(1,0,1)$ parallel to $x+2y+z=0$.

(d) through $(0, 1, 2)$, $(1, 2, 3)$, $(2, 3, 4)$.

9. If the points (x, y, z), $(1, 1, 0)$, and $(1, 2, 1)$ lie on a plane through the origin, what determinant is zero? What equation does this give for the plane?

2 Complex numbers

1. Plot in the complex plane: $z = 2 + i$, its complex conjugate $\overline{z}$, their product $\overline{z}z$ and their sum $z + \overline{z}$.
2. Plot in the complex plane: $\cos(3\pi/4) + \sin(3\pi/4)i$, its square and its cube.
3. Find the two roots of $z^2 - 4z + 5 = 0$.
4. Find the 10 roots of $z^{10} = 2^{10}$.
5. True or false:
   (a) If $z_1 + z_2$ is real, then the $z_i' s$ are complex conjugates.
   (b) If $|z_1| = 2$ and $|z_2| = 4$ then $|z_1 z_2| = 8$.
   (c) If $z = e^{2+9\mathbf{i}}$ then $|z| = e^9$.
   (d) The complex conjugate of $e^{2+9\mathbf{i}}$ is $e^{2-9\mathbf{i}}$.

3 Vector functions and space curves

1. Sketch the curve with parametric equations $x = t, y = t^3$. Find the velocity vector and the speed at $t = 1$.

2. On the circle $x = \cos t$, $y = \sin t$, explain by the chain rule why $dy/dx = -\cot t$.

3. Find parametric equations to go around the unit circle so that the speed at time t is e^t. Start at $x = 1, y = 0$. When is the circle completed?

4. Which vector functions trace the the same curve as $R = t\mathbf{i} + t^2\mathbf{j}$? The speed along the path may be different.
   (a) $2t\mathbf{i} + 2t^2\mathbf{j}$
   (b) $2t\mathbf{i} + 4t^2\mathbf{j}$
   (c) $-t\mathbf{i} + t^2\mathbf{j}$
   (d) $t^3\mathbf{i} + t^6\mathbf{j}$.

5. Find the minimum and maximum speed if $x = t + \cos t$, $y = t - \sin t$.

6. Find $x(t), y(t)$ so that the object goes around the circle $(x-1)^2 + (y-3)^2 = 4$ with speed 1.

7. Let $\overline{v}$ be the velocity. Is it always the case that

$$|\frac{d\overline{v}}{dt}| = \frac{d|\overline{v}|}{dt}?$$

(Answer: *No*). Give an example of a motion where it is true and one where it is not.

8. Construct parametric equations for travel on a helix such that the speed at time t is t.

9. Compute the curvature of the curve $y = e^x$ at the point $(1, 0)$.

10. At which point is the curvature of the curve $y = \ln x$ the largest?

11. Find the unit tangent and the unit normal vectors for $x = \cos(t^2), y = \sin(t^2)$.

12. Find the unit tangent and the unit normal vectors for $x = t, y = \ln \cos t$.

13. Find the curvature of the helix $(3\cos t, 3\sin t, 4t)$.

14. If $\kappa = 0$, then the velocity and acceleration are ____________.

15. Find the tangential and normal components of the acceleration for
    (a) $x = 5\cos t, y = 5\sin t, z = 0$ (circle)
    (b) $x = 1 + t, y = 1 + 2t, z = 1 + 3t$ (line)
    (c) $x = t\cos t, y = t\sin t, z = 0$ (spiral)
    (d) $x = e^t \cos t, y = e^t \sin t, z = 0$ (spiral)
    (e) $x = 1, y = t, z = t^2$ (parabola).

16. For the spiral $x = e^t \cos t, y = e^t \sin t, z = 0$, the angle between position and acceleration is constant. Verify this statement and then find this angle.

4 Functions of several variables

1. Draw the level curves for the four functions
    (a) $\sqrt{4 - x^2 - y^2}$
    (b) $2 - \sqrt{x^2 + y^2}$
    (c) $2 - \frac{\sqrt{x^2+y^2}}{2}$
    (d) $1 + e^{-x^2-y^2}$.

    The level curves of all four are ____________. Draw the four curves $f(x, y) = 1$ and rank them by increasing radius.

2. Suppose the level curves are parallel straight lines. Does the graph have to be a plane?

3. Construct a function whose level curve $f = 0$ is in two separate pieces.

4. Construct a function for which $f = 0$ is a circle and $f = 1$ is not.

5. The level surfaces of $F(x, y, z) = x^2 + y^2 + qz^2$ look like (American) footballs when q is ____________ and like basketballs when q is ____________. What do they look like for $q = -1$?

6. (Fun) Draw a contour map of the top of your shoe.

7. (Fun) Draw contour maps of different kinds of hats.

8. Find the domain of:

   (a) $\frac{1}{(x-y)^2}$

   (b) $\frac{y-x}{z-x}$

   (c) $\sqrt{x^2+y^2-z^2}$

   (d) $\ln(x+y+z)$.

9. Find f_x and f_y for:

   (a) $3x - y + x^2y^2$

   (b) $x^3y^2 - e^y$

   (c) $\ln\sqrt{x^2+y^2}$

   (d) y^x

   (e) $\ln(xy)$

   (f) $\arctan(y/x)$.

10. Compute $\frac{\partial f}{\partial x}$ for $f(x,y) = \int_0^{xy} e^{t^2}\, dt$.

11. Find the limit as $(x,y) \to (0,0)$ or show that it does not exist:

    (a) $\sqrt{x^2+y^2}$

    (b) x/y

    (c) $1/(x+y)$

    (d) $xy/(x^2+y^2)$.

12. Can you define $f(0,0)$ so that $f(x,y)$ is continuous at $(0,0)$:

    (a) $f(x,y) = |x| + |y-1|$

    (b) $f(x,y) = (1+x)^y$

    (c) $f(x,y) = x^{1+y}$.

13. Find the normal vector to the surface at P. Use it to find the tangent plane.

    (a) $z = \sqrt{x^2+y^2}$, $P = (0,1,1)$,

    (b) $x + y + z = 17$, $P = (3,4,10)$,

    (c) $x^2 + yr + z^2 = 6$, $P = (1,2,1)$.

    (d) $z = x^y$, $P = (1,1,1)$.

14. If $f(x,y,z) = xyz$, what is df?

15. You invest $P = \$4000$ at $R = 8\%$ (simple interest). Then you make $I = \$320$ per year. If the numbers change by dP and dR what is dI? If the rate drops 0.002 (to 7.8%) what change dP keeps $dI = 0$? After you compute dP, compute the corresponding *actual* change in I and see if it is indeed close to zero.

16. Resistances R_1 and R_2 have parallel resistance R, where $1/R = 1/R_1 + 1/R_2$. Is R more sensitive to ΔR_1 or ΔR_2 if $R_1 = 1$ and $R_2 = 2$?

17. In baseball, the *batting average* is the ratio of *hits* to *outs*:

$$A = \frac{h}{o}.$$

Suppose I have 25 hits and 100 outs, so that my average is $A = 0.250$. At this point, will a hit have a bigger impact on the average or an out?

18. If x and y change by Δx and Δy, find the approximate change $\Delta\theta$ in the angle $\theta = \arctan(y/x)$.

19. Compute ∇f, then $D_{\overline{u}} f$ and then $D_{\overline{u}} f$ at P:

    (a) $f(x, y) = x^2 - y^2 \quad u = (\sqrt{3}/2, 1/2) \quad P = (1, 0)$.
    (b) $f(x, y) = 3x + 4y + 7 \quad u = (3/5, 4/5) \quad P = (0, \pi/2)$.
    (c) $f(x, y) = e^x \cos y \quad u = (0, 1) \quad P = (0, \pi/2)$.
    (d) $f(x, y) =$ Distance to $(0, 3) \quad u = (1, 0) \quad P = (1, 1)$.

20. Let $f(x, y)$ be a (smooth) function and P any point in its domain. True or false:

    (a) There is always a unit vector u at P for which $D_u f(P) = 0$.
    (b) There is a unit vector u for which $D_u f(P) = 1$.

21. Find the direction (unit vector) u in whichf increases fastest at $P = (1, 2)$. How fast?

    (a) $f(x, y) = 3x + 2y$
    (b) $f(x, y) = e^{x-y}$.
    (c) $f(x, y) = \sqrt{5 - x^2 - y^2}$ (careful).

22. I am at the point $(2, 1, 6)$ on the mountain $z = 9 - x - y^2$ and I move in the North–West direction. Am I climbing up or climbing down? (North is the positive x direction and East is the positive y direction.)

23. Around the point $(1, -2)$ the temperature $T = e^{-x^2-y^2}$. Then $\Delta T \approx$ ______$\Delta x +$ ______Δy (fill the blanks). In which direction u does it get hot fastest?

24. Find df/dt using the chain rule:

    (a) $f = x^2 + y^2 \quad x = t, y = t^2$,
    (b) $f = \sqrt{x^2 + y^2} \quad x = t, y = t^2$,
    (c) $f = \ln(x + y) \quad x = e^t, y = e^t$.

25. If a cone grows in height at the rate of 1 inch per minute and in radius at the rate of 2 inches per minute (starting from both height and radius equal to zero), how fast is its volume growing at $t = 3$ minutes?

26. Find all the critical points of the function $f(x, y) = x^3 + y^3 - 3xy$. Determine if they are local maxima, local minima or saddle points.

27. Find the minimum value of $f(x, y) = x^2 + xy + y^2 - x - y$ on the region $x \le 0$.

28. Do the same on the region $y \geq 1$.

29. Do the same on the region $x \leq 0$ and $y \geq 1$.

30. Find the maximum value of $2x + 3y$ on the circle $x^2 + y^2 = 1$. At what point is the maximum achieved?

31. Find the point on the curve $x^6 + y^6 = 2$ that is closest to the origin. Find the one that is farthest.

32. Minimize $f = 3x + y$ subject to $x^2 + 9y^2 = 1$.

33. The temperature T at a point (x, y, z) in space is $400xyz^2$. Find the highest temperature on the surface of the unit sphere $x^2 + y^2 + z^2 = 1$.

34. Find point(s) on the surface $xyz = 1$ closest to the origin.

35. Which triangle maximzes the sum of the sines of the angles? What about cosines?

5 Miscelleneous word problems

1. The temperature at a point (x, y, z) is

$$T(x, y, z) = \frac{100}{x^2 + y^2 + 1}.$$

A bug is flying according to the equation $(x, y, z) = (t, t^2, t^3)$. Distances are in centimeters and time is in seconds. Describe the rate of temperature change experienced by the bug as it passes the point $(1, 1, 1)$. Give two answers: one in degrees per second and the other in degrees per centimeter.

2. As shown in the diagram, a 2-meter length of wire is to be bent into the shape of a pentagon that has an axis of reflective symmetry, and in which two adjacent angles are right. What is the largest area that can be enclosed by such a shape? (Hint: Be careful when you label the figure–some vari able choices work better than others.)

3. United airlines accepts cabin baggage of *linear dimensions* (length plus height plus width) at most 60 inches. What is the maximum volume of a (cuboidal) box that I can carry in the cabin?

4. You want to design a cylindrical cup that can hold 300 ml coffee. To minimize heat loss, you must minimize the surface area. What is the radius and height of the optimal cup? (Ignore the thickness of the walls.)

5. If a train approaches a crossing at 60 mph and a car approaches (at right angles) at 45 mph, how fast are they coming together when they are both 90 miles from the crossing?

6. The base of a rectangular box costs thre times as much per square foot as do the sides and top. Find the dimensions for the most economical box of volume 100.

7. A rectangular box, open at the top, is to hold 256 cubic centimeters. Find the dimensions of the box for which the minimum surface area.

References

[1] *Calculus*, Gilbert Strang, Wellesley–Cambridge Press

[2] *Calculus and analytic geometry*, George B. Thomas, Jr., Addison Wesley