Moduli of Curves (Week 2)

Exercise 1. Let A be an integral domain. Show that every flat A module M is torsion free. (That is, show that multiplication by a nonzero $a \in A$ is an injective map on M.) Show that the converse holds if A is a DVR, but not in general.

Exercise 2. Compute the flat limits $\lim_{t \to 0} X_t$ for the following examples:

1. $X_t = V(X^2 + tY^2 + t^{-1}Z^2) \subset \mathbf{P}^2$.

2. $X_t = \overline{\{[1 : t^a : t^b], [1 : t^b : t^d]\}} \subset \mathbf{P}^2$.

Exercise 3. Let $Q \subset \mathbf{P}^3$ be smooth quadric surface. Let $C \subset Q$ be a smooth curve of type (a, b). Find the tangent space to the Hilbert schemes of subschemes of $\mathbf{P}^3$ at the point corresponding to $[C \subset \mathbf{P}^3]$.

Exercise 4. Show that the Hilbert scheme is smooth at the point corresponding to a rational normal curve of degree n in $\mathbf{P}^n_{\mathbf{C}}$.

Exercise 5. Let T and P be the components of the Hilbert scheme of subschemes of $\mathbf{P}^3_{\mathbf{C}}$ of Hilbert polynomial $3m + 1$ whose generic points correspond to a smooth twisted cubic and a smooth plane cubic union a point, respectively.

1. Compute the dimension of T and P.

2. Show that the Hilbert scheme is smooth at a general point of T and P.

3. Let C_1 be the curve supported on a nodal plane cubic with an embedded point at the node not contained in the plane. What is the dimension of the tangent space to the Hilbert scheme at $[C_1]$?

4. Let C_2 be a ‘spatial triple line’, namely the scheme defined by the cube of the ideal of a line in $\mathbf{P}^3$. Show that $[C_2]$ lies in T. Moreover, show that $[C_2]$ is a smooth point of T. Hence conclude that $[C_2]$ does not lie in P.

5. Let C_3 be a planar triple line with an embedded point contained in the plane. Show that $[C_3]$ lies in P. Does $[C_3]$ lie in T?