Schemes and functors

Anand Deopurkar

Example 1. Let V be an n dimensional vector space over a field k. The set of one dimensional subspaces of V corresponds bijectively to the points of the projective space $\mathbf{P}V$. More generally, the set of r dimensional subspaces of V corresponds bijectively to the points of the Grassmannian $\mathbf{Gr}(r, V)$.

Example 2. Consider the set of hypersurfaces of degree d in a fixed projective space $\mathbf{P}^n$. Using the homogeneous coordinates $[X_0 : \cdots : X_n]$, we can describe a hypersurface by an equation

$$\sum_{i_1+\cdots+i_n=d} a_{i_1,\ldots,i_n} X_0^{i_1} \cdots X_n^{i_n}.$$

So a hypersurface can be specified by a system of $\binom{n+d}{d}$ coefficients $a_{i_1,\ldots,i_d}$, subject to the restriction that not all of them are zero, and with the understanding that scaling all of them by the same non-zero constant gives the same hypersurface. Therefore, the set of hypersurfaces of degree d in $\mathbf{P}^n$ corresponds bijectively to the points of $\mathbf{P}^N$, where $N = \binom{n+d}{d}$.

Example 3. Let C be a smooth projective curve of genus g over $\mathbf{C}$. Consider the set of line bundles of degree zero on C. A line bundle may be specified by an open cover $\{U_i\}$ of C, and transition functions $g_{i,j} : U_i \cap U_j \to \mathbf{C}^*$ satisfying $g_{i,j} \circ g_{j,k} = g_{i,k}$ on $U_i \cap U_j \cap U_k$. A function $U_i \cap U_j \to \mathbf{C}^*$ is simply an element of O_C^* on $U_i \cap U_j$. In this way, a line bundle gives a Cěch 1-cocycle for the sheaf O_C^* on $\{U_i\}$. It is easy to check that the two cocycles obtained from two isomorphic line bundles differ only by a Cěch coboundary. We thus get a map from the set of line bundles on C to $H^1(C, O_C^*)$.

On the other hand, given an element of $H^1(C, O_C^*)$, we may represent it by a Cěch cocycle $g_{i,j}$ on some covering $\{U_i\}$ of C. Using the $g_{i,j}$ as transition functions, we can then construct a line bundle. It is easy to check that two cocycles that differ by a coboundary give isomorphic line bundles. We thus get a map from $H^1(C, O_C^*)$ to the set of line bundles on C.

The two maps constructed above are mutual inverses. So we may identify the set of line bundles on C with $H^1(C, O_C^*)$.

Now consider the exponential exact sequence of analytic sheaves on C

$$0 \to \mathbf{Z} \to O_C \to O_C^* \to 0,$$

where $O_C \to O_C^*$ is given by $f \mapsto \exp(2\pi i f)$. The induced map $H^1(C, O_C^*) \to H^2(C, \mathbf{Z}) = \mathbf{Z}$ is the degree map. The line bundles of degree zero, therefore, correspond bijectively to the quotient $H^1(C, O_C)/H^1(C, \mathbf{Z})$.

We have $H^1(C, O_C) \cong \mathbf{C}^g$ and $H^1(C, \mathbf{Z}) \cong \mathbf{Z}^{2g}$. Furthermore, $H^1(C, \mathbf{Z}) \subset H^1(C, O_C)$ exhibits $H^1(C, \mathbf{Z})$ as a lattice in $H^1(C, O_C)$. As a result, the quotient is topologically a torus $(S^1)^{2g}$. By construction, it is also a complex manifold. It turns out that it actually has the structure of an algebraic variety.

In any case, the set of line bundles of degree zero on C corresponds bijectively to the points of a (topological) torus, or a complex manifold, or (taking the last statement on faith) a complex algebraic variety.

The examples above show that many sets of algebro-geometric objects are in bijection with points of an algebraic variety. In some sense, such a variety *parametrize* those algebro-geometric objects. We often say that it is the *moduli space* of those objects.

Let us take the example of a Grassmannian. We want to say that the Grassmannian $\mathbf{Gr}(r, n)$ is the moduli space of r dimensional subspaces of an n dimensional space. To give content to this statement, we must define our terms.

Definition 4 (Attempt 1)**.** The moduli space of r dimensional subspaces of an n dimensional vector space is a scheme G whose k-points are in bijection with the set of r dimensional subspaces of k^n.

This definition is almost useless. Many schemes satisfy this definition (so, in particular, our article "the" is grossly misplaced.) Indeed, if we take $k = \mathbf{C}$, then the set of points of any non-finite scheme over $\mathbf{C}$ is in some bijection with the set of r dimensional subspaces of k^n, simply because these are two sets of the same cardinality.

One way to inject some content into the definition is to remember that k points of a scheme are just maps from $\operatorname{Spec} k$ to the scheme. We then look at maps not just from $\operatorname{Spec} k$ but also from other schemes X. Given a map $\phi : X \to \mathbf{Gr}(r, n)$, the subspaces $\phi(x)$ give us a family of r-dimensional subspaces of an n dimensional, parametrized by the points of X. If ϕ is an algebraic map, then the resulting family of subspaces ought to be varying algebraically (whatever that means). Conversely, given an algebraically varying family of r-dimensional subspaces of an n dimensional vector space parametrized by X, we ought to get an algebraic map $X \to \mathbf{Gr}(r, n)$. It is easy to formalize the notion of an algebraically varying family of r-dimensional subspaces of V parametrized by X. Such a family should simply be an (algebraic) sub vector bundle of rank r of the trivial vector bundle O_X^n. We now upgrade our previous attempted definition.

Denote by $\operatorname{Sub}_{r,n}(X)$ the set of rank r sub vector bundles of O_X^n.

Remark 1. The notion of a sub vector bundle of a vector bundle is slightly tricky. A sub vector bundle of O_X^n is *not the same* as a locally free subsheaf of O_X^n. For the correct notion of a sub vector bundle, convince yourself of the following. Let E be a locally free sheaf and $F \subset E$ a subsheaf. Then the following are equivalent.

1. for every $x \in X$, the map on the fibers $F|_x \to E|_x$ is injective.
2. F and E/F are locally free.

We say that $F \subset E$ is a sub vector bundle if it satisfies these conditions.

Definition 5 (Attempt 2)**.** The moduli space of r dimensional subspaces of an n dimensional vector space is a scheme G for which we have a bijection $i_X\colon \mathrm{Maps}(X,G) \to \mathrm{Sub}_{r,n}(X)$ for all schemes X.

This appears better, but on a second thought we realize that we are requiring bijections between (many pairs of) two sets, which are very likely of the same size. However, we do not want arbitrary bijections; we want bijections that are compatible with morphisms. Indeed, suppose we have a map $\phi : X \to Y$. Then the correspondence $i_X\colon \mathrm{Maps}(X,G) \to \mathrm{Sub}_{r,n}(X)$ and $i_Y\colon \mathrm{Maps}(Y,G) \to \mathrm{Sub}_{r,n}(Y)$ must be such that

$$i_X(f \circ \phi) = \phi^* i_Y(f). \tag{1}$$

In other words, the family of subspaces on X obtained from the map $f \circ \phi$ must be the pullback (via ϕ) of the family on Y obtained from f.

Said more formally, we define a contravariant functor from the category Schemes to the category Sets by the rule that sends a scheme X to the set $\mathrm{Maps}(X,G)$ and a morphism $\phi : X \to Y$ to the function $\mathrm{Maps}(Y,G) \to \mathrm{Maps}(X,G)$ defined by $f \mapsto \phi \circ f$. Similarly, we define a functor $\mathrm{Sub}_{r,n}(-)$ by the rule that sends a scheme X to $\mathrm{Sub}_{r,n}(X)$ and a morphism $\phi : X \to Y$ to the pullback map $\phi^*\colon \mathrm{Sub}_{r,n}(Y) \to \mathrm{Sub}_{r,n}(X)$. Saying that there exist bijections $i_X\colon \mathrm{Maps}(X,G) \to \mathrm{Sub}_{r,n}(X)$ for all X that satisfy (1) is the same as saying that there exists a natural isomorphism between the two functors $\mathrm{Maps}(-,G)$ and $\mathrm{Sub}_{r,n}(-)$.

Definition 6. The moduli space of r dimensional subspaces of an n dimensional space is a scheme G such that the functor $\mathrm{Maps}(-,G)$ is naturally isomorphic to the functor $\mathrm{Sub}_{r,n}(-)$.

Having defined our terms, we can state our theorem.

Theorem 7. *There exists a scheme* $\mathbf{Gr}(r,n)$ *with a natural isomorphism of functors*

$$\mathrm{Maps}(-,\mathbf{Gr}(r,n)) \cong \mathrm{Sub}_{r,n}(-).$$

The above procedure illustrates how we will formulate the claim that a certain scheme is the moduli space of a certain class of objects.

Definition 8. Let C be a category. If a contravariant functor $F : \underline{C} \to \underline{\mathrm{Sets}}$ is isomorphic to the functor $\mathrm{Maps}(-,X)$ for some object X of C, then we say that X *represents* F.

Yoneda's lemma guarrantees that if a representing object X exists, then it is unique.

Proposition 9 (Yoneda's lemma)**.** *Let C be any category, X an object of C, and $F : \underline{C} \to \underline{Sets}$ a contravariant functor.*

1. *There is a bijection*

$$\{\textit{Natural transformations from } \mathrm{Maps}(-,X) \textit{ to } F\} \longleftrightarrow F(X).$$

2. *In particular, if Y is another object of $\underline{C}$, then there is a bijection*

$$\{\textit{Natural transformations from } \mathrm{Maps}(-,X) \textit{ to } \mathrm{Maps}(-,Y)\} \longleftrightarrow \mathrm{Maps}(X,Y).$$

In particular, a natural isomorphism from $\mathrm{Maps}(-,X)$ *to* $\mathrm{Maps}(-,Y)$ *gives an isomorphism from X to Y.*

Let us now prove Theorem 7. We will phrase the proof in the language that emphasizes the functorial point of view. For brevity, we will denote the functor $\mathrm{Maps}(-,Y)$ by h_Y. It is often called the *functor of points* of Y (because when we put $\mathrm{Spec}R$ in the place of $-$, we get the set of R-valued points of Y).

Definition 10. Let F be a contravariant functor from $\underline{\text{Schemes}}$ to $\underline{\text{Sets}}$. We say that F is a *sheaf* (in the Zariski topology), if for every scheme X the following holds: for every open cover $\{U_i\}$ of X, and a collection of elements $\alpha_i \in F(U_i)$ that agree on the overlaps $U_i \cap U_j$ (that is, the restriction of α_i and α_j to $U_i \cap U_j$ are equal), there is a unique $\alpha \in F(X)$ that restricts to α_i on U_i.

A functor of the form $h_Y = \mathrm{Maps}(-,Y)$ clearly satisfies the sheaf condition. Indeed, a map $X \to Y$ is uniquely specified by specifying it on an open cover, compatibly on the overlaps. So, first and foremost, a representable functor must necessarily be a sheaf.

Secondly, a scheme is covered locally by affine schemes. We now extend the notion of an open cover to a functor. To this end, we first generalize the construction of fiber products to functors.

Definition 11. Let $f: F \to H$ and $g: G \to H$ be natural transformations between functors from a category to $\underline{\text{Sets}}$. Define the *fiber product* $F \times_H G$ by

$$F \times_H G(S) = F(S) \times_{H(S)} G(S) = \{(a,b) \mid a \in F(S), b \in G(s), \text{ and } f(a) = g(b)\}.$$

Notice that when F, G, and H are representable, then $F \times_H G$ is also representable, and the representing scheme is the usual fiber product.

Definition 12. Let $f: F \to G$ be a map between functors from $\underline{\text{Schemes}}$ to $\underline{\text{Sets}}$. We say that f is an open immersion if for every scheme X and map $h_X \to G$, the fibered product $h_X \times_G F$ has the form h_Y for some Y and the map $Y \to X$ given by $h_X \times_G F \to h_X$ is an open immersion.

We say that a collection of maps $\{F_i \to G\}$ is an *open covering* if each $F_i \to G$ is an open immersion and for every field K, the map $\bigcup F_i(K) \to G(K)$ is surjective.

Proposition 13. *Let F be a contravariant functor from $\underline{\text{Schemes}}$ to $\underline{\text{Sets}}$. Then F is representable if and only if F is a sheaf in the Zariski topology and has an open covering $\{h_{X_i} \to F\}$.*

Proof sketch. Consider $X_{i,j} = X_i \times_F X_j$. Both $X_{i,j} \to X_i$ and $X_{i,j} \to X_j$ are open immersions. Let their images be the open subschemes $X_i^j \subset X_i$ and $X_i^j \subset X_j$. The identifications $X_{i,j} \to X_i^j$ and $X_{i,j} \to X_j^i$ give an isomorphism $\phi_{i,j}: X_j^i \to X_i^j$. One checks that these gluing morphisms satisfy the compatibility $\phi_{i,j} \circ \phi_{j,k} = \phi_{i,k}$ on triple overlaps. So we can glue the X_i along the open sets X_j^i using these isomorphisms and get a scheme X that represents the functor F. $\square$

Proposition 14. *The Grassmannian functor* $\mathrm{Sub}_{r,n}$ *is a sheaf in the Zariski topology.*

Proof. Exercise. $\square$

Finally, we must produce an open cover of the Grassmannian functor. For a subset $I \subset \{1,\dots,n\}$ of size r, define the functor $\mathrm{Sub}^I_{r,n}$ by

$$\mathrm{Sub}^I_{r,n}(X) = \{\text{Sub-bundles } F_i \subset O^n_X \text{ such that the projection } F_i \to O^I_X \text{ is an iso.}\}$$

Here $O^n_X \to O^I_X$ is the projection onto the r coordinates given by I.

Proposition 15. *The collection* $\{\mathrm{Sub}^I_{r,n} \to \mathrm{Sub}_{r,n}\}_I$ *is an open cover.*

Proof. Let us first show that $\mathrm{Sub}^I_{r,n} \to \mathrm{Sub}_{r,n}$ is an open immersion. Consider a map $h_X \to \mathrm{Sub}_{r,n}$. By Yoneda's lemma, such a map is equivalent to an element $[F \subset O^n_X]$ of $\mathrm{Sub}^I_{r,n}(X)$. We have to compute the fiber product $H^I = X \times_{\mathrm{Sub}_{r,n})} \mathrm{Sub}^I_{r,n}$. A map from Y to this fiber product is equivalent to an element of

$$h_X(Y) \times_{\mathrm{Sub}_{r,n}(Y)} \mathrm{Sub}^I_{r,n}(Y),$$

namely to a map $\phi: Y \to X$ and an element $[F_Y \subset O^n_Y]$ of $\mathrm{Sub}^I_{r,n}(Y)$ such that the two elements of $\mathrm{Sub}_{r,n}(Y)$ obtained from ϕ and $[F_Y \subset O^n_Y]$ are the same. Since $[F_Y \subset O^n_Y]$ is an element of $\mathrm{Sub}^I_{r,n}(Y)$, the projection $F_Y \to O^I_Y$ is an isomorphism. The element of $\mathrm{Sub}_{r,n}(Y)$ obtained from $[F_Y \subset O^n_Y]$ is just $[F_Y \subset O^n_Y]$. The element of $\mathrm{Sub}_{r,n}(Y)$ obtained from ϕ is the pullback $[\phi^*F \subset O^n_Y]$. We can thus rephrase the fiber product as

$$H^I(Y) = \{\phi: Y \to X \mid \phi^*F \to O^I_Y \text{ is an isomorphism.}\}.$$

Consider the projection $F \to O^I_X$. The set $x \in X$ such that $F|_x \to O^I_X|_x$ is an isomorphism is an open subset U^I of X (why?). Then

$$H^I(Y) = \{\phi: Y \to X \mid \phi(Y) \subset X\} = \{\phi: Y \to U^I\}.$$

Thus H^I is represented by U^I.

Showing that the collection $\mathrm{Sub}^I_{r,n}$ covers $\mathrm{Sub}_{r,n}$ is easy. $\square$

We finish the construction of the Grassmannian by showing that the subfunctors $\mathrm{Sub}^I_{r,n}$ are representable.

Proposition 16. *The subfunctors* $\mathrm{Sub}^I_{r,n}$ *are represented by* $\mathbf{A}^{r(n-r)}$.

Proof. Let X be a scheme and an element $[F \subset O^n_X$ of $\mathrm{Sub}^I_{r,n}(X)$. Then, by definition, the projection $\pi: F \to O^I_X$ is an isomorphism. Let $i: O^I_X \to O^n_X$ be the standard inclusion. Consider the map $\phi = \pi^{-1} - i: O^I_X \to O^n_X$. Then its projection onto O^I_X is zero, and hence it lands in $O^{I^c}_X$. Thus, from $X \to \mathrm{Sub}^I_{r,n}$, we get a map between the two trivial vector bundles $O^I_X \to O^{I^c}_X$. Such a map is given by an $r \times (n-r)$ matrix, and hence is the same as a map $X \to \mathbf{A}^{r(n-r)}$.

Conversely, given a map $X \to \mathbf{A}^{r(n-r)}$, we interpret it as an $r \times (n-r)$ matrix with entries in O_X, and hence as a map $\phi : O_X^I \to O_X^{I^c}$. We let $F = O_X^I$ and construct an inclusion $F \to O_X^n = O_X^I \oplus O_X^{I^c}$ by

$$v \mapsto (v, \phi(v)).$$

The resulting $[F \subset O_X^n]$ gives a map $X \to \mathrm{Sub}_{r,n}^I$.

It is easy to verify that these two constructions are mutually inverse and give a natural bijection between $\mathrm{Maps}(X, \mathbf{A}^{r(n-r)})$ and $\mathrm{Sub}_{r,n}^I(X)$. Thus, $\mathrm{Sub}_{r,n}^I$ is represented by $\mathbf{A}^{r(n-r)}$. □

Our proof of Theorem 7 produced a Zariski open cover of $\mathbf{Gr}(r,n)$. Any property of a scheme that is local in the Zariski topology can be checked by checking it on the open cover. For example, we see immediately that $\mathbf{Gr}(r,n)$ is smooth of dimension $r(n-r)$. But how can we get more information? It turns out that a lot of properties of the representing scheme can be read off from the functor itself. We end by mentioning two examples, one global and one local.

Let F be a functor represented by a scheme X of finite type. The following proposition gives a criterion for X to be separated/proper in terms of F.

Proposition 17 (Valuative criteria of separatedness and properness)**.** *X is separated (resp. proper) if and only if for every DVR R with fraction field K, the map $F(\mathrm{Spec}\, K) \to F(\mathrm{Spec}\, R)$ induced by the inclusion $\mathrm{Spec}\, K \to \mathrm{Spec}\, R$ is an injection (resp. bijection).*

The following proposition recovers the Zariski tangent space of X from the functor F.

Proposition 18. *Let $x \in F(k)$ be a k point of X. Define*

$$T_x F = \{\xi \in F(\mathrm{Spec}\, k[\epsilon]/\epsilon^2) \mid \xi_0 \in F(k) \text{ is } x\},$$

where ξ_0 is the image of ξ in $F(k)$ under the map induced by the inclusion $\mathrm{Spec}\, k \to \mathrm{Spec}\, k[\epsilon]/\epsilon^2$. Then we have a bijection $T_x F \cong T_x X$.

Exercises

1. Show that the following functors are representable and find the scheme that represents them. The functors are described by their action on schemes; their action on morphisms should be easy to infer.
   (a) $A(X) = \Gamma(X, O_X)$ (the set of global regular functions)
   (b) $B(X) = \Gamma(X, O_X)^*$ (the set of invertible global regular functions)
   (c) $C_n(X) = \{f \in \Gamma(X, O_X) \mid f^n = 1\}$.
2. In Proposition 18, can you read off the vector space structure on $T_x F$ from the functor?
3. Let V be a k vector space and $x \in \mathbf{Gr}(r, V)$ be the k point corresponding to an r-dimensional subspace $S \subset V$. Let $Q = S/V$. Using the functorial description of $\mathbf{Gr}(r, V)$ and Proposition 18, construct an isomorphism

$$T_x \mathbf{Gr}(r, V) \cong \mathrm{Hom}(S, Q).$$

4. Use Proposition 17 to prove that $\mathbf{Gr}(r, V)$ is proper.
5. Let V be a vector bundle on a scheme X. Make sense of $\mathbf{Gr}(r, V)$, the relative Grassmannian, as a functor. Then show that it is representable (with minimal work). The result $\mathbf{Gr}(r, V)$ should map to X and its fiber over $x \in X$ should be the Grassmannian $\mathbf{Gr}(r, V|_x)$.
6. Construct the natural transformations between the functors of points of various $\mathbf{P}^n$ that correspond to the Veronese and Segre maps.
7. Construct the natural transformation that corresponds to the Plücker embedding of a Grassmannian.
8. Let $n \geq 1$ be a integer. Consider the functor

$$F(X) = \{S \subset \mathbf{P}^1 \times X \mid S \to X \text{ is finite and flat of degree } n\}.$$

   Show that F is represented by $\mathbf{P}^n$. What if we replace 'flat' by 'etale'?
   To get a sense of the functor, look at the 'geometric points' of the functor, namely the elements of $F(k)$, where k is an algebraically closed field.

Reference for this section [EH00, Chapter VI].

References

[EH00] David Eisenbud and Joe Harris, *The geometry of schemes*, Graduate Texts in Mathematics, vol. 197, Springer-Verlag, New York, 2000. MR 1730819 (2001d:14002)