Moduli of curves - Dec 2

$$\begin{array}{l} \overline{\mathcal{C}}_g \\ \downarrow \pi \\ \overline{M}_g \end{array} \qquad \begin{aligned} \lambda &= c_1(\pi_* \omega) \\ \kappa &= \pi_*(c_1(\omega)^2) \\ \delta &= [\text{Sing curves}]. \end{aligned}$$

Thm: $12\lambda = \kappa + \delta$.

Background: Grothendieck Riemann Roch.

X a nonsingular variety / alg. closed k. ($\mathbb{C}$).

$K(X)$ = Grothendieck group of vector bundles on X

= Free abelian group gen. by $[F]$ modulo
$[F] = [F'] + [F'']$ for every exact seq. of v.b.
$$0 \to F' \to F \to F'' \to 0$$

= Grothendieck group of coh. sheaves on X (only for X nonsing!

$(K(X), +, \otimes)$ is a ring.

There is a ring homomorphism
$$ch: K(X) \to A^*(X) \text{ or } H^*(X) \qquad (\mathbb{Q}\text{-coeff})$$
defined for a line bundle by:
$$ch: [L] \mapsto \exp(c_1(L)) = 1 + c_1 + \frac{c_1^2}{2!} + \frac{c_1^3}{3!} + \cdots$$

and for $\overset{E}{\overset{\|}{(L_1 \oplus \cdots \oplus L_n)}}$ by
$$ch(L_1 \oplus \cdots \oplus L_n) = \sum ch(L_i). \quad \leftarrow \text{symmetric in } c_1(L_i)$$

$\Rightarrow$ can be expressed in terms of $c_1(E), c_2(E), \ldots, c_r(E), \ldots$

Gives the general definition. First few terms:

$$ch(E) = rk(E) + c_1(E) + \frac{c_1^2(E) - 2c_2(E)}{2!} + \cdots$$

$\mathrm{ch}: K(X) \to A^*(X)$ a ring homomorphism.

Now $f: X \to Y$ a map of smooth varieties.

$f^*: K(Y) \to K(X)$ and $A^*(Y) \to A^*(X)$.

$$\begin{array}{ccc} K(Y) & \xrightarrow{ch} & A^*(Y) \\ f^* \downarrow & & \downarrow f^* \\ K(X) & \xrightarrow{ch} & A^*(X) \end{array} \quad \checkmark.$$

For f proper:

Rf_* or $f_!$: $K(X) \to K(Y)$

$$[F] \mapsto \sum (-1)^i [R^i f_* F].$$

Not a ring homomorphism, but a $K(Y)$-module homomorphism:

$$f_*(f^*\alpha \cdot \beta) = \alpha \cdot f_*(\beta).$$

Does not commute with ch.

$$ch(f_* F) \neq f_* ch(F).$$

<u>GRR</u>: Corrects the above.

$Td : (K(X), +) \to (A^*(X), \times)$ gp. hom.

$$[L] \mapsto \frac{c_1(L)}{1-e^{-c_1(L)}} = 1 + c_1(L) + \frac{c_1(L)^2}{12} - \cdots$$

$$[L_1 \oplus \cdots \oplus L_n] \mapsto \prod \left(\frac{c_1(L_i)}{1-e^{-c_1(L_i)}} \right)$$

$$= 1 + \frac{c_1}{2} + \left(\frac{c_1^2 + c_2}{12} \right) + \cdots$$

$Td(X) := Td(T_X)$

<u>Thm</u>: $f: X \to Y$ proper map of smooth k-var.

$$ch(f_* F \cdot Td(Y)) = f_*(ch(F) \cdot Td(X))$$

<u>Example</u>: X = curve, Y = pt.

$$h^0(F) - h^1(F) = f_*\left((rk(F) + \deg(F)) \cdot (1 + (1-g)[pt]) \right).$$

$$= (1-g)\, rk(F) + \deg(F).$$

Pf of Mumford's relation:

Suffices to prove for all curves $B \to \overline{M}_g$ that the pullbacks satisfy $12\lambda = \kappa + \delta$. Furthermore, may take B transverse to the boundary.

$$\begin{array}{ll} \mathcal{C} & \leftarrow \text{smooth surface (Locally } xy - t \text{ near nodes).} \\ \downarrow \pi & \\ B. & \leftarrow \text{smooth curve.} \end{array}$$

$$\lambda = c_1(\pi_* \omega_\pi) = c_1(R\pi_* \omega_\pi)$$

$$ch(R\pi_* \omega) = \pi_*\left(ch(\omega)\cdot Td(\mathcal{C})\, Td^{-1}(B)\right).$$

$$= \pi_*\left(ch(\omega)\cdot Td(T_{C/B})\right).$$

$$= \pi_*\left(1 + c_1(\omega) + \frac{c_1(\omega)^2}{2}\right)\left(1 + \frac{c_1(T)}{2} + \frac{c_1^2(T) + c_2(T)}{12}\right.$$

$$ch(R\pi_*\omega)\cdot Td(B) = \pi_*\left(ch(\omega)\cdot Td(C)\right).$$

$$\Rightarrow ch(R\pi_*\omega) = \pi_*\left(ch(\omega)\cdot Td(T_c - \pi^* T_B)\right).$$

$$T_c \to \pi^* T_B$$

dual: $$0 \to \pi^*\Omega_B \to \Omega_c \to \Omega_{c/B} \to 0.$$

$$Td(T_c - \pi^* T_B) = 1 - \frac{c_1(\Omega_{c/B})}{2} + \frac{c_1^2(\Omega_{c/B}) + c_2(\Omega_{c/B})}{12}.$$

Now: $$0 \to \Omega_{c/B} \to \omega_{c/B} \to \oplus \mathbb{C}_n \to 0$$

$$\Rightarrow c_1(\Omega_{c/B}) = c_1(\omega)$$

$$c_2(\Omega_{c/B}) = [\Delta]. \quad \delta \subset \mathcal{C} \text{ is the sing. locus of } \pi.$$

$$\Rightarrow ch(R\pi_*\omega) = \pi_*\left(1 + c_1(\omega) + \frac{c_1(\omega)^2}{2}\right)\left(1 - \frac{c_1(\omega)}{2} + \frac{c_1(\omega)^2 + c_2(\omega)}{12}\right)$$

$$\lambda = ch_1(R\pi_*\omega) = \frac{\kappa + \delta}{12}.$$

We saw: For a smooth curve C,

$$T_{[C]}\overline{M}_g = H^1(C, T_C).$$

In general for a stable curve C,

$$T_{[C]}\overline{M}_g = \mathrm{Ext}^1(\Omega_C, \mathcal{O}_C)$$
$$= \mathrm{Hom}(\omega_C, \Omega_C)^\vee$$
$$= H^0(\Omega_C \otimes \omega_C^\vee)^\vee.$$

In a one-parameter family.

$$\begin{array}{ccc} \mathcal{C} & & \\ \downarrow \pi & & \\ B & \xrightarrow{\mu} & \overline{M}_g \end{array}$$

$$\mu^* T\overline{M}_g = \pi_*(\Omega_{\mathcal{C}/B} \otimes \omega_{\mathcal{C}/B}^\vee)^\vee.$$

$$\mu^* T^*\overline{M}_g = \pi_*(\Omega_{\mathcal{C}/B} \otimes \omega_{\mathcal{C}/B}^\vee).$$ <u>Check</u>: $R^1\pi_* = 0$.

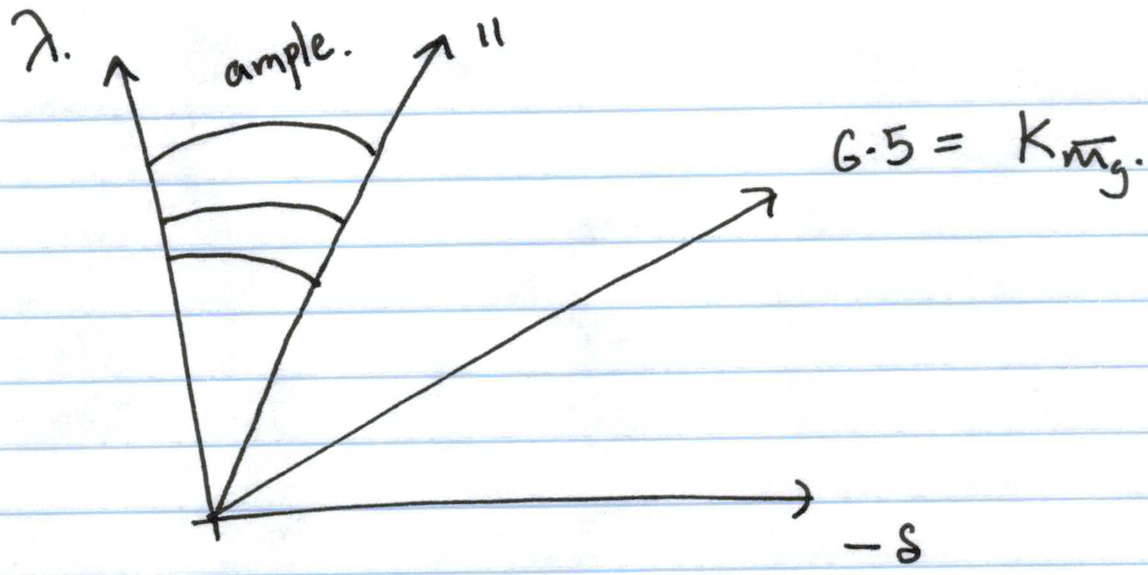
$$ch(R\pi_*) = \pi_*\left(ch(\Omega \otimes \omega^\vee) \cdot \left(1 - \frac{c_1(\omega)}{2} + \frac{c_1(\omega)^2 + [\Delta]}{12}\right)\right).$$

$$= \pi_*\Bigg(\left(1 + c_1(\omega) + \frac{c_1(\omega)^2 - 2[\Delta]}{2}\right) *$$
$$\left(1 - c_1(\omega) + \frac{c_1(\omega)^2}{2}\right) *$$
$$\left(1 - \frac{c_1(\omega)}{2} + \frac{c_1(\omega)^2 + [\Delta]}{12}\right)\Bigg).$$

$$ch_1 = \lambda + \frac{K}{2} - K - \frac{K}{2} + \frac{K}{2} + \frac{K}{2} - \delta$$

$$= \qquad\qquad\qquad K = 12\lambda - \delta$$

$$K_{\overline{M}_g} = \lambda + K - \delta = \boxed{13\lambda - 2\delta.}$$



Thm: $a\lambda - \delta$ is ample iff $a > 11$

Q: What is the full ample cone? Unknown.

Thm: $K_{\bar{M}_g}$ is big if $g \geq 24$.

idea: $K_{\bar{M}_g}$ = ample + effective.
so produce a divisor of slope < 6.5 that is effective

Brill-Noether-divisor (g odd):

Closure of Image of $H_{d,g} \to M_g$ for $d = \frac{g+1}{2}$.

$(2g + 2d - 5 = 2g + g + 1 - 5 = 3g - 4)$.

has class $\left(6 + \frac{12}{g+1}\right)\lambda - \delta - \dots\dots\dots$

$\Rightarrow$ slope < 6.5 for $g \geq 24$.

Q: which linear comb of λ, δ are effective? Unknown.