Stable Reduction (Nov 13)

Let us first prove stable reduction assuming that the general fiber is smooth. More precisely, we will prove the following. (over $\mathbb{C}$)

~~Thm: Let R be a DVR with fraction~~

Thm: Let R be a DVR with frac. field K. Let $C \to \operatorname{spec} K$ be a smooth proper curve. There exists a finite extension K'/K such that $C \times_K K' \to \operatorname{spec} K'$ extends to a stable curve $C' \to \operatorname{spec} R'$, where $R' =$ int. closure of R in K'.

Proof: Set $\Delta = \operatorname{spec} R$, $\Delta^* = \operatorname{spec} K$ etc. Let t be the uniformizer.

Step 0: Extend $C \to \Delta^*$ to any flat proper $X \to \Delta$. ~~By taking the normalization if~~

Step 1: Let $X_1 \to X$ be a resolution of singularities of X (which is a surface, smooth except possibly over $0 \in \Delta$).

Step 2: By blowing up X_1 repeatedly at (smooth) points, arrange so that $(t=0)_{red} \subset X$ is a nodal curve. Call the result $X_2 \to X_1$.
(i.e. normal crossings divisor).

Then, we have the divisor $(t=0) = \sum m_i C_i$
where $m_i \geq 1$, C_i are at worst nodal curves and C_i, C_j intersect transversally for $i \neq j$ (and no three intersect at a pt).

Step 3: Pass to the base change $R[s]/(s^n - t)$ where n is the LCM of the m_i's. Call this new DVR $\tilde{\Delta}$.

Let $X_3 =$ normalization of $\left(X_2 \times_\Delta \tilde{\Delta}\right)$

Then ~~$X_3 \to \tilde{\Delta}$ has a central fiber which is~~ the central fiber of $X_3 \to \tilde{\Delta}$ is a (reduced) nodal curve.

Furthermore, the singularities of X_3 are ~~an~~ A_k singularities ~~of~~ (for various k).

<u>Step 4</u>: Resolve the A_k (minimally) singularities to get $X_4 \to X_3$, where X_4 is non singular. Then the central fiber of $X_4 \to \tilde{\Delta}$ is a prestable curve.

<u>Step 5</u>: Contract the -1 rational curves on X_4. (These are "rational tails")
Contract the -2 rational curves (These are "rational bridges").

The first can be done by Castelnuovo's thm (and the resulting surface is still smooth.) The second can be done by taking the image of the resulting surface under $|K^m|$ for $m \gg 0$, where K is the canonical divisor. (<u>Check</u>: $K = 0$ only on the rational -2 curves in the central fiber and $+$ve on all other curves.)

$\boxed{\text{* all rings should have Spec in front!!}}$

<u>That's it</u>: The assertions in step 3 need justification by local calculation. Here is the calculation: analytically locally on X_2, the map * $X_2 \to \Delta$ has the form $\mathbb{C}[x,y,t]/(t - x^a y^b) \longrightarrow \mathbb{C}[t]$.

Let $a = dp$, $b = dq$ where $\gcd(p,q) = 1$ and $n = dpqr$. for some r.

$X_2 \times_\Delta \tilde{\Delta} \to \tilde{\Delta}$ has the form $\mathbb{C}[x,y,s]/(s^n - x^a y^b) \to \mathbb{C}[s]$.

We normalize (in the ring of total quotients). We can describe the normalization in two stages. First, note the factorization

$$s^{dpqr} - x^{dp} y^{dq} = (s^{pqr})^d - (x^p y^q)^d = \prod_{\xi^d = 1} (s^{pqr} - x^p y^q \cdot \xi)$$

ξ: d^{th} root of unity.

So the first step of the normalization is

$$\mathbb{C}[x,y,s]\big/(s^n - x^a y^b) \longleftarrow \coprod_{\xi} \mathbb{C}[x,y,s]\big/(s^{pqr} - \xi x^p y^q)$$

Secondly, we normalize each piece (wlog $\xi = 1$ piece). The normaliz^n is:

$$\mathbb{C}[x,y,s]\big/(s^{pqr} - x^p y^q) \longleftarrow \mathbb{C}[\alpha,\beta,s]\big/(\alpha\beta - s^r)$$

$$x = \alpha^q$$
$$y = \beta^p.$$

Thus, the local analytic picture of $X_3 \to \tilde{\Delta}$ is

$$\mathbb{C}[\alpha,\beta,s]\big/(\alpha\beta - s^r) \longrightarrow \mathbb{C}[s],$$

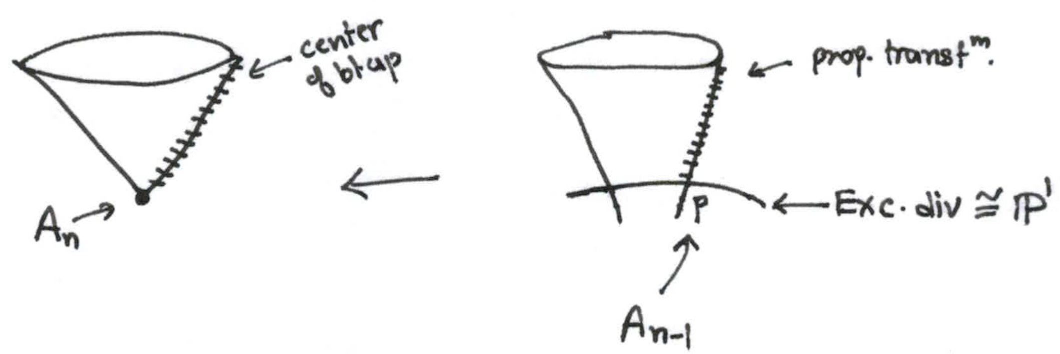
verifying the assertions in step 3.

Remark on resolving A_n sing: $xy - t^{n+1} \subset \mathbb{C}[x,y,t]$

Blow up (x,t)

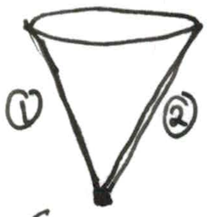
Chart 1: y, u, t ; $x = ut$	Chart 2: y, v, x ; $t = vx$
$(uy - t^n)$. $\leftarrow A_{n-1}$	$y - v^{n+1}x^n$ $\leftarrow$ smooth
Exc div: ~~$u=0$~~, $t = 0$	Exc div: $x = 0$
$\cong \mathbb{A}^1 \cup_p \mathbb{A}^1$ (reduced).	$\cong \mathbb{A}^1$

Picture:



inductively.

· Repeating the process:



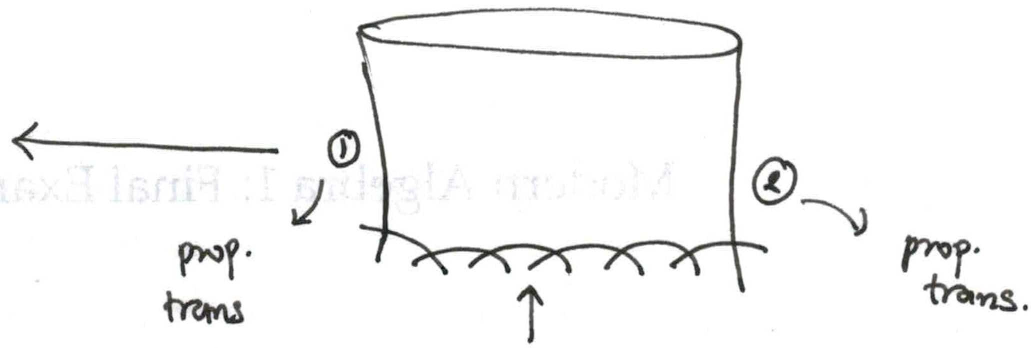
A_n

$\mathbb{C}[x,y,t]/(xy-t^n)$.

①: $x=0,\ t=0$

②: $y=0,\ t=0$

$① \cup ② \equiv (t=0)$



Chain of n $\mathbb{P}^1$'s.

$(t=0) \equiv \underbrace{① \cup \text{chain} \cup ②}_{\text{reduced}}$