Moduli of Curves - Nov. 20

Last time : Deformations of plane affine curves with isolated singularities

$C_0 : f(x,y) = 0 \subset \mathbb{A}^2 \quad / \; k$ alg. closed field.

Let $g_1, \dots, g_r \in k[x,y]$ be a basis of $k[x,y]/(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$.

Set $\Lambda = k[[t_1, \dots, t_r]]$

Consider $C_\Lambda \subset \Lambda[x,y]$ given by

$$f(x,y) + t_1 g_1(x,y) + \dots + t_r g_r(x,y) = 0 \longrightarrow \underset{R}{\overset{\Lambda}{\shortparallel}}$$

Then C_Λ gives a transformation

$$h_R \to \mathrm{Def}_{C_0},$$

given explicitly by the following rule :

$$h_R(A) \to \mathrm{Def}_{C_0}(A)$$

$$(\varphi : R \to A) \mapsto f(x,y) + \varphi(t_1) g_1(x,y) + \dots + \varphi(t_r) g_r(x,y) \subset A[x,y].$$

Thm : $h_R \to \mathrm{Def}_{C_0}$ is versal.

Pf : Consider a small ext^n $0 \to k \xrightarrow{\varepsilon} \tilde{A} \to A \to 0$.

Let $C_{\tilde{A}} \to \tilde{A}$ be a def of C_0 over $\tilde{A}$ such that

$(*)$ $C_A \xrightarrow{\sim} A[x,y]/(f(x,y) + a_1 g_1(x,y) + \dots + a_r g_r(x,y))$

Suppose $C_{\tilde{A}}$ is given by $\tilde{g}(x,y)$ in $\tilde{A}[x,y]$.

We have (from $*$) an iso

$$A[x,y]/g(x,y) \longleftrightarrow A[x,y]/f(x,y) + \sum a_i g_i(x,y).$$

$x \mapsto X$, $y \mapsto Y$. Then $\quad U_0 = 1$, $U \in A[x,y]$ unit.

$$g(X,Y) = U \cdot (f(x,y) + \sum a_i g_i(x,y)).$$

Lift X, Y, U, to $\tilde{A}[x,y]$ and a_i to $\tilde{A}$ arbitrarily.

Then $\tilde{g}(\tilde{X},\tilde{Y}) = \tilde{U}\cdot\left(f(x,y) + \sum \tilde{a}_i\, g_i(x,y)\right) + \varepsilon\cdot \text{error}.$

where $\text{error} \in k[x,y]$.

By our choice of g_i, we can write

$$\text{error} = p_1(x,y)\, f(x,y) + \cancel{\textstyle\sum q_i(x,y)}\; q(x,y)\frac{\partial f}{\partial x} + r(x,y)\frac{\partial f}{\partial y} + \sum \alpha_i\, g_i(x,y)$$

where $\alpha_i \in k$.

Now, observe that

1) $\tilde{g}(\tilde{X}+\varepsilon q, \tilde{Y}+\varepsilon r) = \tilde{g}(\tilde{X},\tilde{Y}) + \varepsilon \frac{\partial f}{\partial x}\cdot q + \varepsilon\cdot\frac{\partial f}{\partial y}\cdot r.$

2) $(\tilde{U}+\varepsilon p)\left(f(x,y) + \sum \tilde{a}_i\, g_i(x,y)\right) = \tilde{U}\left(f(x,y) + \sum \tilde{a}_i\, g_i(x,y)\right) + \varepsilon\, p\cdot f$

3) $\tilde{U}\left(f(x,y) + \sum(\tilde{a}_i + \varepsilon\alpha_i)\, g_i\right) = \tilde{U}\left(f(x,y) + \sum \tilde{a}_i\, g_i(x,y)\right) + \varepsilon \sum \alpha_i\, g_i$

So by making $\tilde{X} \to \tilde{X}+\varepsilon q$, $\tilde{Y} \to \tilde{Y}+\varepsilon r$
$\tilde{U} \to \tilde{U}+\varepsilon p$, $\tilde{a}_i \to \tilde{a}_i + \varepsilon\alpha_i$

we can make the error vanish.

$\square$.

<u>Example</u> ① $f(x,y) = xy$.

$\Rightarrow$ Versal deformation $(xy - t) \subset \Lambda[x,y]$, $\Lambda = k[[t]]$.

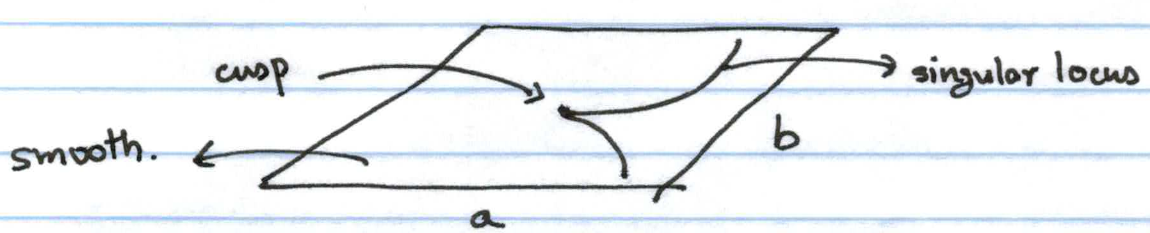
$t=0 \Leftrightarrow$ singular curve (original).

$t\neq 0 \Leftrightarrow$ smooth curve.

② $f(x,y) = y^2 - x^3$ $\quad \frac{\partial f}{\partial x} = 3x^2 \quad \frac{\partial f}{\partial y} = 2y. \quad 1, x, \cancel{xy}$

$\Rightarrow$ versal deformation

$\Lambda = k[[a,b]]$, $\quad y^2 - x^3 - ax - b.$



$4a^2 + 27b^3$

Deformations of smooth affine schemes. X_0.

Thm: Let A be an artin local k-algebra and $X_A \to \operatorname{spec} A$ a deformation of X_0. Then $X_A = X_0 \times_k A$.

Pf. Some preliminary observations:
Let X_0 be any scheme, and X_A, X'_A two deformations of X_0 over A. If there is a map $X_A \to X'_A$, then it is an isomorphism.

Pf: Induct on the length of A. $0 \to k \to A \to \bar{A} \to 0$

The underlying top. space is the same, so everything reduces to the map of the sheaf of rings. We have

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_{X_0} \arrow[r] \arrow[d, equal] & \mathcal{O}_{X_A} \arrow[r] \arrow[d] & \mathcal{O}_{X_{\bar{A}}} \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & \mathcal{O}_{X_0} \arrow[r] & \mathcal{O}_{X'_{A}} \arrow[r] & \mathcal{O}_{X'_{\bar{A}}} \arrow[r] & 0
\end{tikzcd}$$

five lemma $\Rightarrow$ iso.

Pf of thm: We only need to produce a map

$$X_A \to X_0 \times_k A$$

or in fact $X_A \longrightarrow X_0$.

Again, induct on the length of A, and use the following:

Lemma: Let A/k be a smooth k-algebra, and

$$0 \to I \to \tilde{B} \to B \to 0$$

s.t. $I^2 = 0$., (an extension of k-algebra B by an infinitesimal ideal).

Then any map $A \to B$ lifts to $A \to \tilde{B}$.

Pf:

$$\begin{tikzcd}
0 \arrow[r] & J \arrow[r] \arrow[d] & k[x] \arrow[r] \arrow[d, "\varphi"] & A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I \arrow[r] & \tilde{B} \arrow[r] & B \arrow[r] & 0
\end{tikzcd}$$

$J^2 \to 0$. want $J \to 0$. Adjust φ.

$(\varphi_1 - \varphi_2) : k[x] \to I$ is a derivation. Conversely,

$\varphi + \delta : k[x] \to \tilde{B}$ is a homomorphism for any derivation δ.

$$\begin{tikzcd}
0 \arrow[r] & J/J^2 \arrow[r] \arrow[d] & \Omega_{k[x]/k} \arrow[r] \arrow[dl, dashed, "\delta"] & \Omega_{A/k} \arrow[r] & 0 \\
 & I. & & &
\end{tikzcd}$$

$\delta \leftarrow$ exists because is projective.

Change φ to $\varphi + \delta$.

□. ~~Ex.~~

~~Prop. Let X_0 be an affine scheme~~

Prop: X_0 an affine scheme, $\psi : X_A \xrightarrow{\sim} X'_A$. Then the set of isoms $X_{\tilde{A}} \to X'_{\tilde{A}}$ extending ψ is a PHS under ~~$\mathrm{Def}(X_0)$~~, $\mathrm{Hom}(\Omega_{X_0}, \mathcal{O}_{X_0})$. (or empty) $(0 \to k \xrightarrow{\varepsilon} \tilde{A} \to A \to 0)$.

Pf:

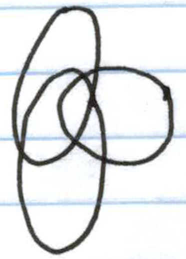
$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r, "\varepsilon"] \arrow[d, equal] & \tilde{B}_1 \arrow[r] \arrow[d, dotted, "\varphi"] & B_1 \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & B \arrow[r, "\varepsilon"] & \tilde{B}_2 \arrow[r] & B_2 \arrow[r] & 0
\end{tikzcd}$$

$\varphi_1 - \varphi_2$ is a derivation of B into B.

□.

Deformations of smooth schemes: $X. = \bigcup U_i$ ↪ affine.

First order :–



← trivial $U_i \times k[\varepsilon]/\varepsilon^2$.

On $U_i \cap U_j$: Glue $\Rightarrow$ $s_{ij} \in \mathrm{Hom}(\Omega_{U_{ij}}, \mathcal{O}_{U_{ij}})$.
subject to agreement on triple over laps.

$\Rightarrow$ First order defs $= H^1(X, T_X)$.

(Obstructions to higher order ext^{n}s in $H^2(X, T_X)$).

Defs of ~~nodal~~ curves.. with isolated sings. (i.e. reduced).

$X_0 =$

U_i open cover s.t. U_i affine
$U_i \cap U_j$ smooth.

Claim: $\mathrm{Def}\, X_0 \to \prod \mathrm{Def}(U_i)$ is smooth.

Pf: Lifting : $\tilde{A} \to A \to 0$ small ext^n.

X_A , $X_A|_{U_i} = U_{i,A}$ given.

$U_{i,\tilde{A}}$ also given. Want to glue extending

$$\begin{tikzcd} U_{i,\tilde{A}}|_j \arrow[r] & U_{j,\tilde{A}}|_i \\ U_{i,A}|_j \arrow[r] \arrow[u, hook] & U_{j,A}|_i \arrow[u, hook] \end{tikzcd}$$

Both $U_{i,\tilde{A}}|_j$ and $U_{j,\tilde{A}}|_i$ are ext'ns of $U_{ij,A}$ and U_{ij} be smooth $\Rightarrow$ both are isomorphic, φ_0!

$\Rightarrow$ Choice of iso (extending previous) is a PHS under $\mathrm{Hom}(\Omega_{U_{ij}}, \mathcal{O}_{U_{ij}})$. Choose one (say φ_{ij}).

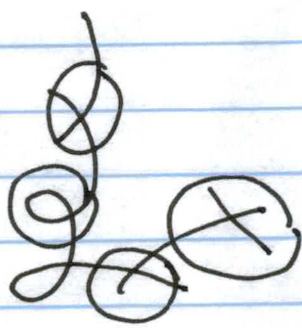
On triple overlaps, $(\varphi_{ik} - \varphi_{ij} \circ \varphi_{jk})$ defines a 2-cocycle of $T_{U_{ij}}$, and ~~if~~ the iso can be fixed iff this is a co-boundary.

but $H^2(X, T_X) = 0$ $\Rightarrow$ can always be fixed.

$\Rightarrow$ Can always glue. $\Rightarrow$ lifting done

$\square$.

<u>Cor</u>: Local structure of $\overline{M}_g$. near

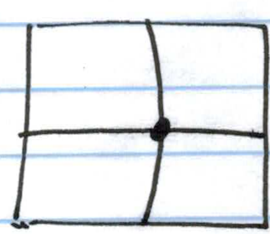


$$\underline{\mathrm{Def}}_{X_0} \xrightarrow{sm} \prod \mathrm{Def}(U_i).$$

trivial if U_i smooth.

$k[[t_i]]$ if U_i a node.

$$\mathrm{Def}_{X_0} \xrightarrow{sm} k[[t_1]] \times k[[t_2]] \times \dots \times k[[t_r]]. \qquad r = \#\text{ nodes}.$$



t_2.

t_1

$$\begin{array}{c} \Delta \subset \overline{M}_g \\ \| \\ \overline{M}_g \setminus M_g \end{array}$$

$\Rightarrow$ $\overline{M}_g$ is smooth and $\Delta \subset \overline{M}_g$ a normal crossings divisor

$\square$.