Moduli of curves : Nov 25

Last time: $\overline{M}_g$ is smooth $/k$ and $\Delta := \overline{M}_g \setminus M_g$ is a divisor with normal crossings.

Today: Line bundles and divisors on $\overline{M}_g$.

$$\begin{tikzcd} C_g \arrow[r, hook] \arrow[d, "\pi"] & \overline{C}_g \arrow[d, "\pi"] \\ M_g . \arrow[r, hook] & \overline{M}_g \end{tikzcd}$$

Recall the Hodge bundle $\quad E = \pi_*(\Omega_{C_g/M_g})$.

Set $\quad \lambda = \det E$.

We want to extend E to all of $\overline{M}_g$. The natural extension of Ω that lets us do this is the relative dualizing sheaf.

Def: X/k a proj. pure n-dim scheme. A dualizing sheaf for X is a sheaf ω_X with a map $t: H^n(X, \omega_X) \to k$ s.t. for every coherent sheaf $\mathcal{F}$ on X, the map

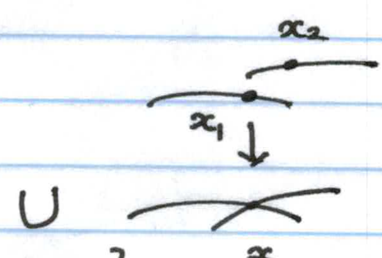
$$\mathrm{Hom}(\mathcal{F}, \omega_X) \times H^n(X, \mathcal{F}) \to H^n(X, \omega_X) \to k$$

is a perfect pairing.

Thm: Dualizing sheaves exist (unique up to unique iso) for Cohen-Macaulay X.
Are invertible if X is LCI (or "Gorenstein")
Coincide with $\Lambda^n \Omega_X$ if X is smooth.

Description for X nodal: Let $\nu: \overline{X} \to X$ be the normalization and $D \subset \overline{X}$ the preimage of the nodes.

$$\omega_X \subset \nu_* \omega_{\overline{X}}(D)$$



$$\omega_X|_U = \{ \omega \in \Gamma(\nu^{-1}(U), \omega_{\overline{X}}(D)) \text{ s.t. } \mathrm{res}_{x_1}\omega + \mathrm{res}_{x_2}\omega = 0 \}$$

In analytic local coordinates

$\mathbb{C}[x] \oplus \mathbb{C}[y]$.

$\mathbb{C}[x,y]/xy$.

$$\omega_{\bar{X}}(D) = \mathbb{C}[x]\langle \tfrac{dx}{x} \rangle \oplus \mathbb{C}[y]\langle \tfrac{dy}{y} \rangle. \qquad \mathrm{Res}\langle \tfrac{dx}{x} \rangle = 1$$

$$\mathrm{Res}\langle \tfrac{dy}{y} \rangle = 1.$$

$$\omega_X = \mathbb{C}[x,y]/(xy) \left\langle \tfrac{dx}{x} - \tfrac{dy}{y} \right\rangle.$$

$$x \cdot \left(\tfrac{dx}{x} - \tfrac{dy}{y}\right) = dx$$

$$y \cdot \left(\tfrac{dx}{x} - \tfrac{dy}{y}\right) = -dy.$$

Note: $\nu^* \omega_X = \omega_{\bar{X}}(D)$.

Also: we have

$$0 \to \Omega_X \to \omega_X \to \oplus\, \mathbb{C}_{\text{nodes}} \to 0 \qquad (*).$$

By duality: $H^0(X, \omega_X) \cong H^1(X, \mathcal{O}_X)^\vee \leftarrow$ g-dim. vector space.

More generally, given a family $\mathcal{X} \xrightarrow{\pi} B$ of nodal curves, there exists a relative dualizing sheaf ω_π satisfying the relative version of Serre duality, in particular

$$\pi_*(\omega_\pi) \cong R^1\pi_*(\mathcal{O}_{\mathcal{X}})^\vee =: E \qquad \underline{\text{Hodge bundle}}.$$

Also, relative version of $(*)$ holds

$$0 \to \Omega_{\mathcal{X}/B} \to \omega_\pi \to Q \to 0 \qquad Q \text{ supp on } \mathrm{Sing}(\pi).$$

In any case, we get a line bundle $\lambda := \det E$ on $\overline{M}_g$.

Since $\Delta \subset \overline{M}_g$ is a divisor, and $\overline{M}_g$ is smooth, $\mathcal{O}(\Delta)$ is a line bundle on $\overline{M}_g$ (dual of the ideal sheaf $\mathcal{I}_\Delta$).

Next, we'll define a codimension 1 Chow class.

(Rem: For smooth ~~proj~~ DM stacks, chow groups / rings with $\mathbb{Q}$ coefficients can be defined in the usual way, and they coincide with the corresponding obj. for the coarse spaces.).

The Kappa class:

$$\begin{array}{c} \overline{\mathcal{C}}_g \\ \downarrow \pi \\ \overline{\mathcal{M}}_g \end{array} \qquad \begin{array}{l} c_1(\omega_\pi) \in A^1(\overline{\mathcal{C}}_g, \mathbb{Q}). \\ c_1(\omega_\pi)^2 \in A^2(\overline{\mathcal{C}}_g, \mathbb{Q}). \end{array}$$

$$\kappa := \pi_*\left(c_1(\omega_\pi)^2 [\overline{\mathcal{C}}_g]\right)$$

Example. $\mathbb{P}^1 \xrightarrow{\mu} \overline{\mathcal{M}}_g$ given by a pencil of plane curves of deg d.
$[s:t] \longmapsto sF + tG$, where F, G are deg d poly in X,Y,Z (fixed and general).

Then $g = \binom{d-1}{2}$.

$$\begin{array}{ccc} \mathcal{C} & \subset & \mathbb{P}^1 \times \mathbb{P}^2 \\ \downarrow & \swarrow \pi & \\ \mathbb{P}^1 & & \end{array} \qquad \mathcal{C} = V(sF + tG).$$
$= $ divisor of type $(1,d)$.

Let us compute $\mu^*\lambda$, $\mu^*\delta$, $\mu^*\kappa$.

$$0 \to \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(-1,-d) \to \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2} \to \mathcal{O}_C \to 0 \qquad \pi_*.$$

$$0 \to \mathcal{O}_{\mathbb{P}^1} \xrightarrow{\sim} \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}(-1) \otimes H^1(\mathbb{P}^2, \mathcal{O}(-d)) \to 0.$$
$$\to R^1\pi_*\mathcal{O}_C \to \mathcal{O}_{\mathbb{P}^1}(-1) \otimes H^2(\mathbb{P}^2, \mathcal{O}(-d)) \to 0.$$

$$\Rightarrow E^\vee = \mathcal{O}_{\mathbb{P}^1}(-1) \otimes H^0(\mathbb{P}^2, \mathcal{O}(d-3))^\vee.$$
$$\cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \qquad (g \text{ times}).$$

$$\Rightarrow \lambda = \det(E^\vee) = g. = (d-1)(d-2)/2$$

$\alpha = (1,0)$ on $\mathbb{P}^1 \times \mathbb{P}^2$
$\beta = (0,1)$

K: $\omega_{\mathcal{C}/\mathbb{P}^1} = K_{\mathcal{C}} \otimes \pi^* K_{\mathbb{P}^1}^{\vee}$

$$= (-2\alpha - 3\beta) + (\alpha + d\beta)\big|_{\mathcal{C}} - (-2\alpha)\big|_{\mathcal{C}}$$

$$= (\alpha + (d-3)\beta)\big|_{\mathcal{C}}.$$

$$
\begin{aligned}
c_1(\omega_{\mathcal{C}/\mathbb{P}^1})^2 &= (\alpha + (d-3)\beta)^2 \cdot [\mathcal{C}] \\
&= (\alpha + (d-3)\beta)^2 \cdot (\alpha + d\beta) \\
&= (2(d-3)\alpha\beta + (d-3)^2\beta^2)(\alpha + d\beta) \\
&= 2d(d-3) + (d-3)^2 \\
&= (d-3)(3)(d-1).
\end{aligned}
$$

$$
\begin{aligned}
\delta &= \#\text{ singular fibers} \\
&= 3(d-1)^2.
\end{aligned}
$$

<u>Thm</u>: (Mumford's relation). In $A^2(\overline{M}_g, \mathbb{Q})$, we have
$$12\lambda = \kappa + \delta.$$

<u>Check</u>:
$$
\begin{aligned}
6(d-1)(d-2) &= 3(d-3)(d-1) + 3(d-1)^2 \\
&= 3(d-1)(d-3+d-1) \\
&= 6(d-1)(d-2). \quad \checkmark.
\end{aligned}
$$

<u>Pf</u>: Suffices to prove that $12\lambda = \kappa + \delta$ on any curve $B \to \overline{M}_g$. Furthermore, may assume that B is smooth, generically lands in M_g and lies transverse to the boundary.

$$
\begin{array}{ccc}
\mathcal{C} & \hookrightarrow \text{ locally } \sout{\text{over}} \text{ at a node} & (xy-t) \\
\downarrow & & \uparrow \\
B. & \Rightarrow \mathcal{C} \text{ a smooth surface.} & k[t].
\end{array}
$$

$$\lambda = \det(\pi_* \omega)$$
$$= c_1(\pi_* \omega).$$

$$0 \to \Omega_\pi \to \omega_\pi \to \oplus\, \mathbb{C}_{\text{nodes}} \to 0$$

Grothendieck-Riemann-Roch:

$$ch(R\pi_* \omega) = \pi_*\left(ch(\omega) \cdot Td_\pi\right)$$

$$= \pi_*\left(\left(1 + \omega + \frac{\omega^2}{2}\right)\left(1 - \frac{c_1(\Omega_\pi)}{2} + \frac{c_1(\Omega_\pi)^2 + c_2(\Omega_\pi)}{12}\right)\right).$$

$$c_1(\Omega_\pi) = c_1(\omega_\pi) = \omega.$$
$$c_2(\Omega_\pi) = \delta.$$

$$\Rightarrow \quad c_1(R\pi_*\omega) = c_1(\pi_*\omega)$$
$$= \pi_*\left(1 + \omega + \frac{\omega^2}{2}\right)\left(1 - \frac{\omega}{2} + \frac{\omega^2 + \delta}{12}\right).$$

$$\Rightarrow \quad \lambda = \pi_* \frac{(\omega^2 + \delta)}{12} = \frac{\kappa + \delta}{12}.$$

$\square$.