Moduli of curves Nov 6

$\mathfrak{X}/S$ a DM stack, $U \to \mathfrak{X}$ atlas $R = U \times_{\mathfrak{X}} U$,

Then $R \rightrightarrows U$ a groupoid.

$$\{\text{Q-coh sheaves on } \mathfrak{X}\} \underset{\text{eqv.}}{\longleftrightarrow} \{\text{Q-coh sheaves on } [R \overset{s}{\underset{t}{\rightrightarrows}} U]\}$$

$$\| $$

$$(\mathcal{F} \text{ on } U,\ \psi: s^*\mathcal{F} \to t^*\mathcal{F},\ \text{cocycle})$$

Examples (1) $\mathfrak{X} = BG$, $U = \cdot \to BG$

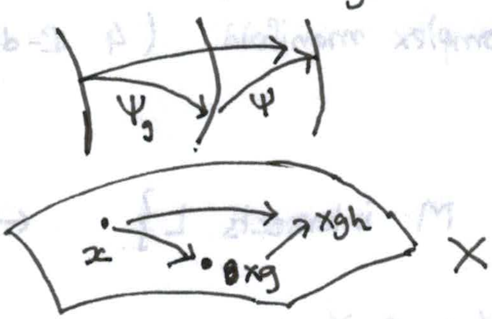
Then Q-coh sheaf on $\mathfrak{X}$ $=$ Representation of G

$\cup$ coh sheaf $=$ $\cup$ finite dim rep.

(2) $\mathfrak{X} = [X/G]$, $U = X \to [X/G]$ $R = G \times X \overset{p}{\underset{a}{\rightrightarrows}} X$.

Q-coh sheaf on $\mathfrak{X}$ $=$ Sheaf on X, $\mathcal{F}$, iso $\psi: p^*\mathcal{F} \to a^*\mathcal{F}$

At (g,x): $\mathcal{F}_x \xrightarrow{\psi} \mathcal{F}_{x.g}$



$=$ "G-linearized sheaf on X."

(3) $\mathrm{Pic}(\mathcal{M}_{1,1}) = \mathbb{Z}/12\mathbb{Z}$.

Line bundle on $\mathcal{M}_{11}$:

① For every $\begin{array}{c} E \\ \downarrow \pi \\ S \end{array}$ $\rightsquigarrow$ L_E on S.

② $\begin{array}{ccc} E_S & \to & E_T \\ \downarrow & \square & \downarrow \\ S & \xrightarrow{f} & T \end{array}$ $\rightsquigarrow$ iso $\psi: f^*L_T \xrightarrow{\sim} L_S$. + compatibility

Example – presentations for BG
X/G

q.coh. sheaves on BG
X/G.

Hodge bundle on M_g.

det of Hodge bundle λ.

~~Harer's~~

Example: $\mathrm{Pic}\,(M_{1,1}) = \mathbb{Z}/12\mathbb{Z}$. $\quad / \mathbb{C}$.

Pf: Let $\mathcal{L}$ be a line bundle on $M_{1,1}$.

$$\begin{tikzcd} (E,p) \arrow[r, "\text{inv.}"] \arrow[d] & (E,p) \arrow[d] \\ (E,p) \arrow[r, equal] & \operatorname{spec} \mathbb{C} \end{tikzcd} \rightsquigarrow \quad L|_t \xrightarrow{\pm 1} L|_t \rightsquigarrow \text{elt of } \mathbb{Z}/2\mathbb{Z}. \quad \alpha.$$

$E_1 : (y^2 = x(x-1)(x+1), \infty)$

$$\begin{tikzcd} C \arrow[r, "\sim"] \arrow[d] & C \arrow[d] \\ {} \arrow[r, phantom, "="] & {} \end{tikzcd}$$

$x \mapsto -x$
$y \mapsto iy$. $\quad \leftarrow$ order 4.
square = ~~hyperell~~ (-1 map).

$\mathcal{L} \rightsquigarrow \beta \in \mathbb{Z}/4\mathbb{Z}$ reducing to α.

$E_2 : \quad y^2 = (x-1)(x-\omega)(x-\omega^2)$.

$y \mapsto -y$ $\qquad$ order 6
$x \mapsto \omega x$. $\qquad$ cube = hyperell. inv.

$\Rightarrow \mathcal{L} \rightsquigarrow \gamma \in \mathbb{Z}/6\mathbb{Z}$. reducing to α

$(\alpha, \beta, \gamma) \in \mathbb{Z}/6 \times_{\mathbb{Z}/2} \mathbb{Z}/4 = \mathbb{Z}/12$.

get $\underline{\mathrm{Pic}}\,(M_{1,1}) \to \mathbb{Z}/12$.

Surjectivity –

Hodge bundle $\Lambda|_{(E,p)} = H^0(E, \omega_E)$. ← one dim

$y^2 = (x)\cdot(x-1)(x-\lambda)$

$\omega = \dfrac{dx}{y}. \longrightarrow i\left(\dfrac{dx}{y}\right).$

$y^2 = x\,(x-\omega)\,(x-\omega^2)$

$\dfrac{dx}{y} \rightsquigarrow -\omega\left(\dfrac{dx}{y}\right).$

$\Rightarrow$ $\mathrm{Pic}\,(M_{1,1}) \longrightarrow \mathbb{Z}/12\mathbb{Z}$ is surjective.

Injectivity: Suppose $\mathcal{L}$ is a line bundle on $M_{1,1}$ s.t. $\beta_{\mathcal{L}}, \gamma_{\mathcal{L}} = 1$.

Want: $\mathcal{L} \cong \mathcal{O}$.

$$
\begin{tikzcd}
\mathrm{Isom}(\lambda,\lambda). \arrow[dr, "2:1"] \arrow[rrr, "f_0"] \arrow[dddr, "g_0"'] & & & \lambda\text{-line}. \arrow[ddd, "\text{étale}"] \arrow[ddddr, bend left, "S_3 \text{ quotient}."] & \\
& S_3\times\lambda \arrow[urr, "f_1"] \arrow[dr] \arrow[dd, "g_1"] & & & \\
& & \lambda\times_j\lambda \arrow[uur, "f_2"'] \arrow[dl, "g_2"] & & \\
& \lambda. \arrow[rr] \arrow[drrr] & & M_{1,1}. \arrow[dr] & \\
& & & & j\text{-line}.
\end{tikzcd}
$$

Given : ~~Descent datum:~~ $\cancel{\Psi}$ $\cancel{F}$

Line bundle L on λ-line :

$\Psi : f^*L \longrightarrow g^*L.$

$L^{\circlearrowleft\sigma}_{(E,p)} : \longrightarrow L_{(E,p)}$

Claim 1 : $\alpha = 1 \Rightarrow \Psi$ descends to an iso. $f_1^*L \to g_1^*L.$

Claim 2 : $\beta, \gamma = 1 \Rightarrow \Psi$ descends to an iso $f_2^*L \longrightarrow g_2^*L.$

Concl : Ψ is a descent datum for a line bundle on the j-line.

But $\mathrm{Pic}\,(j\text{-line}) = \mathrm{triv} \Rightarrow \Psi$ describes the trivial line bundle.

Another Proof:

$c(g,t)$ a descent datum on $S_3 \times \lambda$.

Consider: $j\text{-line} \setminus 0, 1728. = j^\circ$

then
$$\begin{tikzcd} S_3 \times \lambda^\circ \arrow[r] \arrow[d] & \lambda^\circ \arrow[d] \\ \lambda^\circ \arrow[r] & j^\circ \end{tikzcd}$$
étale S_3 quotient

$\Rightarrow$ $c(g,t)$ restricted to $S_3 \times \lambda^\circ$ gives a descent datum for a line bundle on j° $\Rightarrow$ must be trivial.

$\Rightarrow$ $\exists$ function $u(t)$ on λ° such that:

$$c(g,t) = \frac{u(gt)}{u(t)}.$$

Issue: $u(t)$ may have zeros or poles on $\lambda \setminus \lambda^\circ = \left\{ \begin{array}{l} 3 \text{ pts over } 1728 \\ 2 \text{ pts over } 0 \end{array} \right\}$.

Near a pt over 0: stab $\cong \mathbb{Z}/3\mathbb{Z} \ni \sigma$

~~Let~~ $u(t) \cdot t^a$ ~~is a local parameter.~~ is holomorphic and nonzero for some a.

but then $c(g,t) = \dfrac{u(gt)}{u(t)}$ $\Rightarrow$ $c(\sigma, 0) = 1$ $\Rightarrow$ $a \equiv 0 \pmod 3$.

Similarly zero or pole at the other point $\equiv 0 \pmod 2$.

These can be taken care of by modifying $u(t)$ by the pull back of an appropriate function on the j-line!

6. (10 points) Prove that a if a prime p divides the order of a group G, then G contains an element of order p.

Another proof:

Every elliptic curve can be written as
$$y^2 = x^3 + ax + b. \quad = E_{a,b}$$

Furthermore, if we let $t \in \mathbb{G}_m$ act by $t: a \mapsto t^4 a$, $b \mapsto t^6 b$, then

$E_{a,b} \xrightarrow{\sim} E_{t^4 a, t^6 b}$

$y^2 = x^3 + ax + b \xrightarrow{\sim} y^2 = x^3 + t^4 a x + t^6 b$

$y \mapsto t^3 y$
$x \mapsto t^2 x.$

Also $E_{a,b}$ smooth
$\Updownarrow$
$\Delta := 27a^2 + 4b^3 \neq 0$

Prop: We have $[\mathbb{A}^2 \setminus \Delta / \mathbb{G}_m] \xrightarrow{\sim} M_{1,1}$.

Pf sketch: ~~Rig~~ (Left → Right):

$T \to [\mathbb{A}^2 \setminus \Delta / \mathbb{G}_m] \quad \leftrightarrow \quad \begin{array}{c} P \xrightarrow{a,b} \mathbb{A}^1 \\ \downarrow \\ T \end{array} \qquad \begin{array}{l} a(tx) = t^4 a(x) \\ b(tx) = t^6 b(x). \end{array}$

$\Updownarrow$

$(T, L, \underset{H^0(\ )}{a \in L^4}, \underset{H^0(\ )}{b \in L^6})$ a,b global sections.

Given $(T, L, a, b) \rightsquigarrow \begin{array}{c} \mathbb{P}(L^2 \oplus O),\ O(1). \\ \downarrow \pi \\ T \end{array}$ Let $B = O(4) \otimes \pi^* L^{-2}$.

~~Construct $\sigma \in H^0(O(4))$~~ $\ = \ $ ~~$H^0(L^8 + L^6 + L^4 + L^2 + O)$~~ ~~$b\ a\ 0\ 1$~~

Construct $\sigma \in H^0(B) = H^0(\underset{b}{L^6} + \underset{a}{L^4} + \underset{0}{L^2} + \underset{1}{O} + \underset{0}{L^{-2}})$

The data $(O(2) \otimes L^{-1}, \sigma)$ defines a double cover of $\mathbb{P}$. This is the required elliptic curve.

(Right → Left): Exercise.

Let us use this to compute the Picard group.

If time permits.

- Quotient description of M_g.
- ~~Kot~~ Keel-Mori, GIT — approaches to coarse space.
- Definition of separated / proper.