Moduli of Curves:

Last time: M_g is an irreducible quasiprojective variety.

Today: ① M_g is neither projective nor affine $(g \geq 3)$

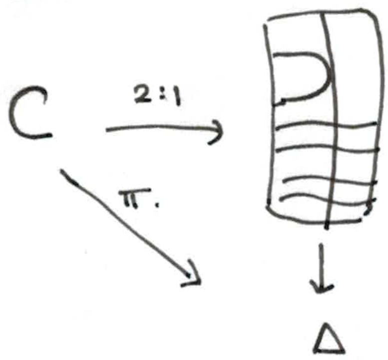
② Cohomology / chow ring of M_g

③ Tautological ring of M_g.

Obs: M_g is not proper.

Pf: Construct $\Delta^* \to M_g$ that does not extend to $\Delta \to M_g$.

$B \subset \mathbb{P}^1_\Delta$ a divisor of deg $2g+2$ over Δ, étale over Δ^*, simply branched over Δ.



$\pi : C|_{\Delta^*} \to \Delta^*$ smooth

$\Delta^* \to M_g$.

$C_0 =$ 

Claim 0: C_0 is not smooth.

Claim 1: There is no C'/Δ sm s.t. $C'/_{\Delta^*} \to \Delta^*$ is isomorphic to C/Δ^*. Indeed, if there were, then we have a birational map

$C' \dashrightarrow C$ between smooth surfaces.

$\exists$ $\tilde{C} \to C'$ a sequence of blow ups on C'_0 and a birational morphism $\tilde{C} \to C$, which is a seq. of blow downs on $\tilde{C}_0$.

Excercise: This is impossible.

However, this does not imply that $\Delta^* \to M_g$ does not extend. After all, not all maps come from families.

Fact: Given $\Delta \to M_g$ $\exists$ finite cover $\tilde{\Delta} \to \Delta$ s.t. $\tilde{\Delta} \to \mathrm{Fun}(M_g)$.

Example.

$$\begin{array}{ccccc} \tilde{\Delta} & \longrightarrow & \lambda\text{-line} = \mathbb{A}^1 \setminus 0,1 & \leftarrow & \text{carries a family.} \\ \downarrow & & \downarrow \text{finite} & & \\ \Delta & \longrightarrow & j\text{-line} = \mathbb{A}^1 & & \end{array}$$

This picture generalizes.

① Local: $p \in M_g$ corresponding to C, let $G = \mathrm{Aut}(C)$.

Then $\exists$ étale neighborhood $U \ni p$ of M_g s.t.

$$\text{finite map} \longrightarrow \begin{tikzcd} \tilde U \circlearrowleft G \arrow[d] \arrow[dd, bend right=60, "/G"'] & \\ F|_U \arrow[r] \arrow[d] & \mathrm{Fun}(M_g) \arrow[d] \\ U \arrow[r] & M_g \end{tikzcd}$$

② Global: Rigidify $\mathrm{Fun}(M_g)$.

Ex. $M_g[n] = \{C \text{ sm proj} + \text{a symplectic basis of } H_1(C, \mathbb{Z}/n\mathbb{Z})\}$.

$= $ moduli of curves with a level n structure.

For $n \geq 3$, (C, level-n structure) have no nontrivial automorphisms.

As a result $M_g[n]$ is represented by a quasi-proj. variety

Global cover
$$\begin{tikzcd} M_g[n] \arrow[d] \arrow[dd, bend left=60, "\text{finite}"] \\ \mathrm{Fun}(M_g) \arrow[d] \\ M_g. \end{tikzcd}$$

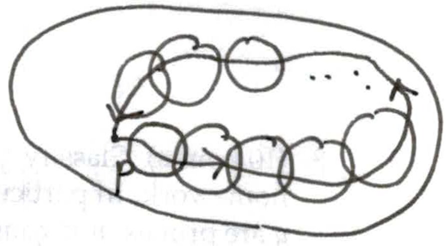
Exercise: Show that after any base change $\tilde\Delta \to \Delta$, the family $C \times_{\Delta^*} \tilde\Delta^* \to \tilde\Delta^*$ does not extend to a smooth family over $\tilde\Delta$ by modifying the argument before (i.e. using the description of birational maps between surfaces.)

We'll see a purely topological way of seeing this. The key is the following result from differential topology.

Thm: Let $\mathfrak{X} \xrightarrow{\pi} U$ be a smooth proper map between two manifolds. Then π is locally a fibration. i.e. $\exists$ open cover $\{U_i\}$ of U s.t. $\mathfrak{X}|_{U_i} \cong X \times U_i$ over U_i

$\hookrightarrow$ diffeomorphic. (Ehresmann's lemma).

Monodromy on $H_*(X_p, \mathbb{Z})$, $p \in U$.

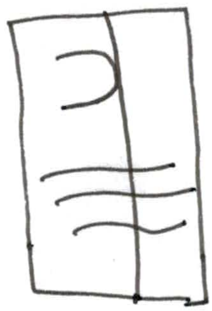


$$m_\gamma : H_*(X_p, \mathbb{Z}) \longrightarrow H_*(X_p, \mathbb{Z}).$$

"parallel transport"

$$m : \pi_1(U, p) \to GL(H_*(X_p, \mathbb{Z})).$$

Let us compute the monodromy in our family.



$\mathbb{P}^1 \times \Delta \supset B$.

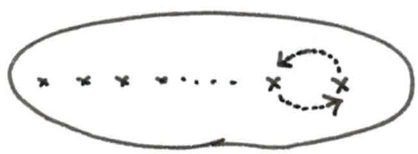
$\downarrow$

Δ.

C_t

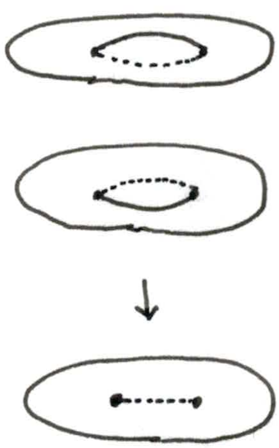
$\downarrow$ 2:1 cover.

P:



Want to transport homology classes

Focus around the two points (classes supported away from them don't move.)



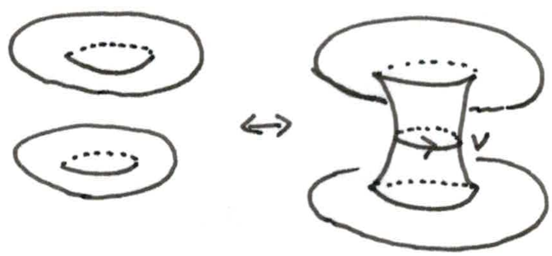
$(-\sqrt{t}, t)$

$(\sqrt{t}, t)$

t

Reflect in x axis

$\longleftrightarrow$

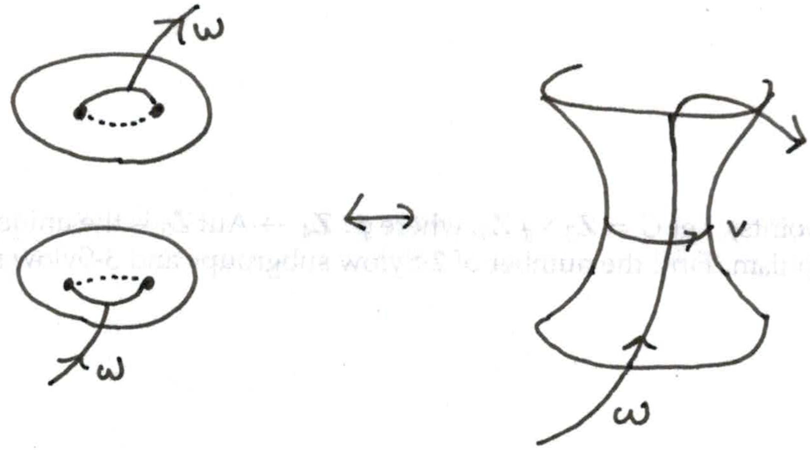


consider v = lift of :

v = "vanishing cycle."

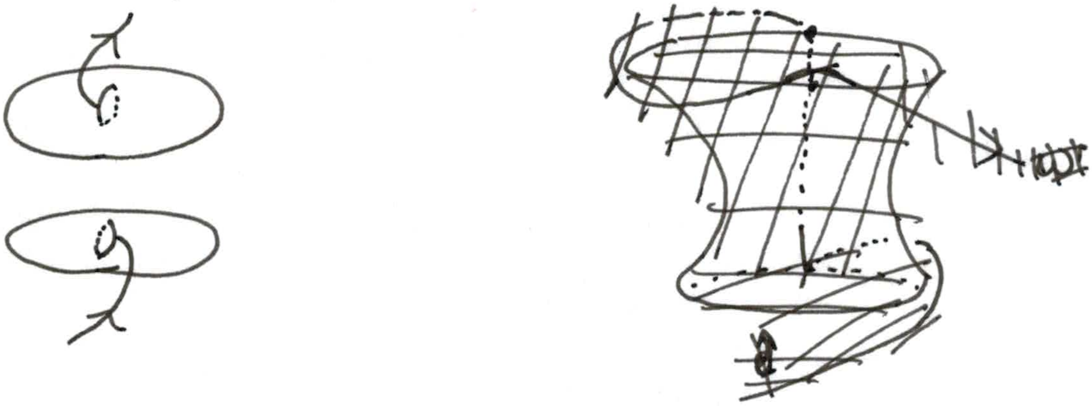
Consider ω

Initial:

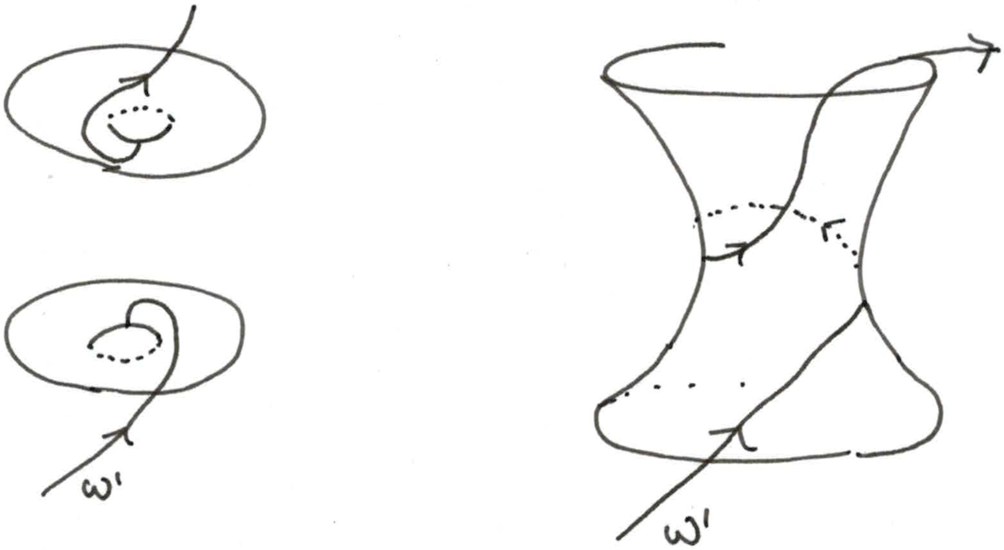


Now go in a loop in Δ^*

Intermed:



Final:



That is $\omega' = \omega + v$.

In general: $m_1 : x \longmapsto x + (x, v)\, v$.

$$\begin{array}{c|cc} & v & \omega \\ \hline v & 1 & 1 \\ \omega & 0 & 1 \end{array}$$

Observe: No power of m_1 is trivial.

$\Rightarrow$ $C|_{\Delta^*} \to \Delta^*$ does not extend to a smooth family even after a base change.

Complete curves in M_g: (Kodaira construction)

Let C be a curve of genus $g \geq 2$.

Consider $H_{3,b=1}(C) = \left\{ f: D \to C \;\middle|\; \begin{array}{l} \deg f = 3 \text{ and } f \text{ is} \\ \text{totally branched at one} \\ \text{point in } C \end{array} \right\}$

$$\begin{array}{c} H_{3,b=1}(C) \\ \text{finite map.} \downarrow \text{br.} \\ C \end{array}$$

$\dim H_{3,b=1}(C) = 1$ and we have

projective. $\rightarrow$ $H_{3,b=1}(C) \longrightarrow M_g$.

($g = 3h-2$ by Riemann Hurwitz).

$\Rightarrow$ M_g has a complete curve for any g of the form $3h-2$ $(h \geq 2)$.

Prop.: M_g has a complete curve for all $g \geq 3$.

In fact $\exists$ complete curve in M_g passing through any finite number of given points.

Thm (Diaz): There does not exist a complete $(g-1)$ dim subvar. of M_g.

Rem. ~~In practice,~~ The bound is expected to be far from sharp. Iterating the above construction. gives complete subvar. of dim about $\log_3(g)$, which is very far from $(g-2)$.

Q (open): What is the max. dim of a complete subvar. of M_g?

open also for M_4 (I think), definitely for M_5.

($\exists$ curves, but do $\exists$ surfaces)?