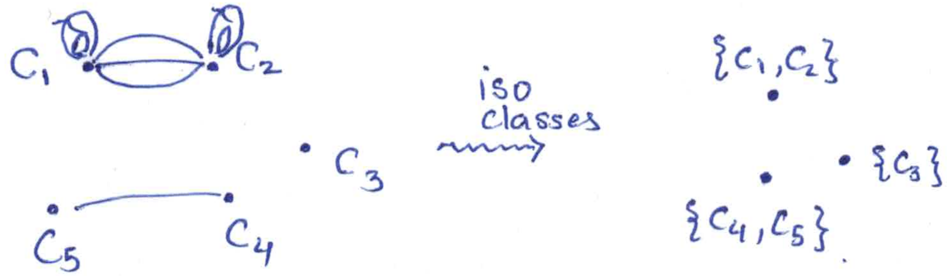
Moduli of curves - Oct 21

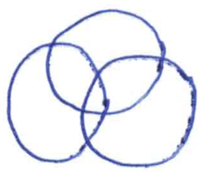
Recall our first attempt at formulating M_g as a functor:

M_g : Schemes $\rightsquigarrow$ Sets

$$S \longmapsto \left\{ \begin{array}{l} \pi : C \to S \\ \text{sm prop. curve} \end{array} \right\} \Big/ \text{iso}$$



Loss of information comes to bite us while gluing -



$$C_i \qquad C_j$$

$$C_i|_{U_{ij}} \;—\; C_j|_{U_{ij}}$$

may be a choice involved here.

make choices $\rightsquigarrow$ ① choices don't agree on triple overlaps
② too many choices agree on triple overlaps.

Result - Not a sheaf.

Motivation for "stacks" - Generalize the notion of sheaves to accommodate objects as above.

Recall Scheme = Sheaf + locally Spec R

Likewise Algebraic Stack = Stack + locally Spec R

First condition → maps can be locally defined and "glued."

Second condition → locally, a map from X to our object corresponds to a number of regular functions on X with polynomial conditions.

"algebraic"

Def: A groupoid is a category where every arrow is an isom.

Rem: ① A groupoid with one object "is" a group.

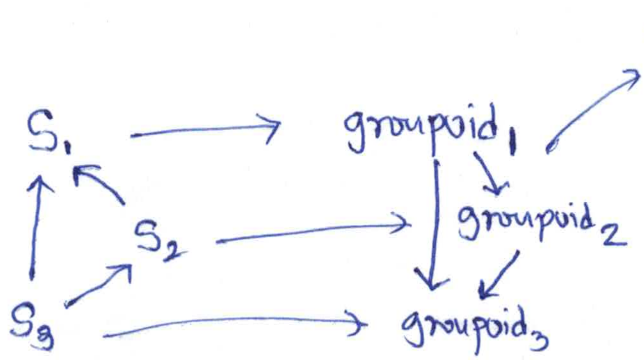
② A groupoid with a unique arrow between any pair of objects "is" a set.

③ Eqv. a groupoid with trivial "automorphism groups" is a set.

aut. grp.

Recall: A sheaf = contravar. set valued functor + gluing cond.

Stack = "groupoid-valued functor" + gluing cond.



may not be equal.

Leads to slightly messy definitions.

Alternatively:

Def: Let S be a category. A category fibered in groupoids over S is a category F with a functor $p: F \to S$ such that

①
$$\begin{tikzcd} \exists X_2 \arrow[r] \arrow[d, dashed, "p"] \arrow[dr, dashed, "p"] & X_1 \arrow[d, dashed, "p"] \\ B_2 \arrow[r] & B_1 \end{tikzcd}$$

$$\begin{tikzcd} F \arrow[d, "p"] \\ S \end{tikzcd}$$

②
$$\begin{tikzcd} X_3 \arrow[rr] \arrow[dr, "\exists !"'] & & X_1 \\ & X_2 \arrow[ur] & \end{tikzcd}$$

$$\begin{tikzcd} B_3 \arrow[rr, bend left] \arrow[r] & B_2 \arrow[r] & B_1 \end{tikzcd}$$

Rem ①

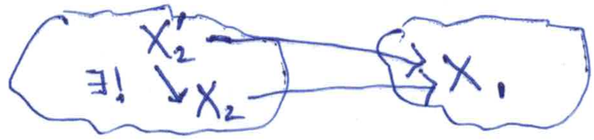
$$\begin{tikzcd} X_2 \arrow[r, "f"] \arrow[d, dashed] & X_1 \arrow[d, dashed] \\ B_2 \arrow[r, "s"] & B_1 \end{tikzcd} \quad : F$$

Then f is an iso. iff s is an iso.

② Given $B \in S$, consider

$F(B)$ = Category whose objects are $X \in F$ with $p(X) = B$ and morphisms are $X_1 \xrightarrow{f} X_2$ s.t. $p(f) = id_B$.

Then $F(B)$ is a groupoid

③ $F(B_2)$. 

$F(B_1)$ ← groupoids

$$B_2 \longrightarrow B_1$$

So we guarrantee the existence of "pullbacks" which are unique up to a unique iso. without insisting on a particular one.

Examples ① If $F : S^{op} \to Sets$ is a functor, then we can make it a CFG : obj = (s, w), $s \in S$, $w \in F(s)$

maps : for every $f : s \to t$ put
$(s, w) \to (t, w')$ if $w = f^* w'$.

In particular, every scheme gives a CFG.

② : $\underline{M_g}$: A CFG over $\underline{\text{Schemes}}$.

$\underline{\text{Obj}}$ = $(\pi : C \to S)$

$\underline{\text{maps}}$ = Pull back diagrams

$$\begin{tikzcd} C_1 \arrow[r] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\square"] & C_2 \arrow[d, "\pi_2"] \\ S_1 \arrow[r] & S_2 \end{tikzcd}.$$

③ $\underline{\text{Vect}_n}$: A CFG over $\underline{\text{Schemes}}$.

$\underline{\text{Obj}}$ = $(S, \mathcal{E})$, $\mathcal{E}$ is a vect. bundle of rank n over S

$\underline{\text{maps}}$ = $S_1 \xrightarrow{f} S_2$ + an iso $\mathcal{E}_1 \xrightarrow{\sim} f^*\mathcal{E}_2$

Similarly $\underline{\text{Coh}}$, $\underline{\text{Qcoh}}$., ~~in particular $\underline{\text{Pic}}$ = Vect_1~~

④ G a group scheme (over S).

$\underline{BG}$: CFG over Schemes.

$\underline{\text{Objects}}$: $(\pi_S : P \to T)$ a principal G-bundle

$\underline{\text{maps}}$: Pullback diagrams

$$\begin{tikzcd} P_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & P_2 \arrow[d] \\ T_1 \arrow[r] & T_2 \end{tikzcd}$$

⑤ G a group acting on X.

$[X/G]$: CFG over Schemes

Objects : $(\pi : P \to T,\ f : P \to X)$ π a Principal G-bundle
f a G-eqv. map

$\underline{\text{maps}}$: Pullbacks

$$\begin{tikzcd} P_1 \arrow[r] \arrow[d] \arrow[rr, bend left] & P_2 \arrow[r] \arrow[d] & X \\ T_1 \arrow[r] & T_2 & \end{tikzcd}$$

commuting with maps to X.

⑥ $\underline{\mathcal{C}_g}$: "Universal curve"

<u>Objects</u> – $(\pi : C \to S, \ \sigma : S \to C)$
sm curve   section

<u>maps</u> :

$$\begin{tikzcd}
C_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & C_2 \arrow[d] \\
T_1 \arrow[r] \arrow[u, bend left, "\sigma_1"] & T_2 \arrow[u, bend right, "\sigma_2"']
\end{tikzcd}$$

<u>Def</u>: F_1 and F_2 two CFG's fibered over S.
A <u>map</u> of CFG's is a functor $f: F_1 \to F_2$ s.t.

$$\begin{tikzcd}
F_1 \arrow[r, "f"] \arrow[d, "P_1"'] \arrow[dr, phantom, "\circlearrowleft"] & F_2 \arrow[d, "P_2"] \\
S \arrow[r, equal] & S
\end{tikzcd}$$
i.e. $P_1 = P_2 \circ f$.

<u>Ex</u>: ① $\mathcal{C}_g \to \mathcal{M}_g$
② $[X/G] \to BG$
③ $X \to [X/G]$
④ morphisms of schemes $\leftrightarrow$ maps of associated CFG's.