~~Deligne-Mumford~~ Stacks:

Oct 23

C a category. We defined a CFG over C as a generalization of the notion of a contravariant functor to $\underline{\text{Sets}}$.

In particular, $F: C^{op} \to \underline{\text{Sets}}$ gives

$\underline{F}$ = Obj : (A, ω) $A \in \text{ob}\, C$ $\omega \in F(A)$

maps : $\omega_A \xleftarrow{F(f)} \omega_B$

$$A \xrightarrow[f]{} B \quad \text{in } C$$

In particular, given $X \in \text{ob}\, C$, we get the functor $\text{Maps}(-, X)$, and thus $\underline{X}$, a fibered category. Concretely:

$\underline{X}$: Obj : $(A, f: A \to X)$ mor:
$$\begin{tikzcd} & & X \\ A \arrow[rru, bend left, "f"] \arrow[r] & B \arrow[ru, "g"'] & \end{tikzcd}$$

Lemma (Yoneda): Consider a CFG $p: F \to C$. Then $\text{Hom}(\underline{X}, F)$ is a category (objects = ~~functors~~ maps of CFG's, morphisms = Nat. transf).

Then $\text{Hom}(\underline{X}, F) \xrightarrow{\sim} F(X)$ given by

$p \longmapsto p(X, \text{id})$ is an equivalence.

Definition of a Stack – Recall sheaf = functor + gluing conditions.

Likewise, a stack will be a CFG with gluing conditions. To describe the gluing conditions, it is better to think of a CFG as a "groupoid valued functor" than a category. That is, make choices of pull backs, denoted by upper $*$.

$$\begin{tikzcd} f^*X \arrow[r] \arrow[d, dotted] & X \arrow[d, dotted, "p"] \\ A \arrow[r, "f"] & B \end{tikzcd} \quad \text{in } C.$$

Also, given $X \to Y \in F(B)$

$$\begin{tikzcd} & f^*Y \arrow[rr] & & Y \\ f^*X \arrow[ru, "\exists!"] \arrow[rr] & & X \arrow[ru, "\alpha"'] & \end{tikzcd}$$

$\exists!$ $f^*X \to f^*Y$

denote by $f^*\alpha$.

$$A \xrightarrow{f} B$$

Then $f^*: F(B) \to F(A)$ is a functor.

$F(B)$ is a groupoid, $F(A)$ is a groupoid.

Now assume that we have a (Grothendieck) topology on C, i.e. a notion of when a map $U \to A$ is a covering.

<u>Examples</u> ① (surjective), Zariski covers.
② étale covers
③ smooth covers
④ flat (and finite presentation) covers.
etc. for $C = $ <u>schemes</u>.

<u>Def</u>: A CFG $p: F \to C$ is a <u>stack</u> if the following two conditions hold

① <u>Descent for morphisms</u>: For every covering $U \xrightarrow{\pi} A$ and $\alpha, \beta \in F(A)$ if we are given $\tilde{f}: \pi^*\alpha \to \pi^*\beta$ such that $pr_1^*(\tilde{f}) = pr_2^*(\tilde{f})$ $(U \times_A U \rightrightarrows U)$, then $\exists!\ f: \alpha \to \beta$ s.t. $\tilde{f} = \pi^* f$

② <u>Descent for objects</u>: For every covering $U \xrightarrow{\pi} A$, if we are given $\tilde{\alpha} \in F(U)$ and $g: pr_1^*\tilde{\alpha} \to pr_2^*\tilde{\alpha}$ such that $pr_{12}^*(g) \circ pr_{23}^*(g) = pr_{13}^*(g)$, there exists $\alpha \in F(A)$ ~~such that~~ along with $\iota: \pi^*\alpha \to \tilde{\alpha}$ s.t.

$$\begin{tikzcd}
pr_1^*\pi^*\alpha \arrow[r, "pr_1^*\iota"] \arrow[d, equal] & pr_1^*\tilde{\alpha} \arrow[d, "g"] \\
pr_2^*\pi^*\alpha \arrow[r, "pr_2^*\iota"] & pr_2^*\tilde{\alpha}
\end{tikzcd}$$

<u>Example</u>: Suppose $F = \underline{F}$ for a functor $F: C^{op} \to$ <u>Sets</u>.

Then $F(A)$: $\circlearrowleft^{id}\ \circlearrowleft^{id}\ \circlearrowleft^{id}\ \circlearrowleft^{id}$ So

①: Given $\tilde{f}: \pi^*\alpha \to \pi^*\beta \iff \left.\begin{array}{l}\pi^*\alpha = \pi^*\beta \\ \alpha = \beta\end{array}\right\}$ conclusion i.e. F is a "separated presheaf"
the condition on $U \times_A U$ is vacuous.
(gluing is unique, if exists)

② Given $\tilde{\alpha}$. The existence of $g \iff pr_1^*\tilde{\alpha} = pr_2^*\tilde{\alpha}$ $\Big\}$ Conclusion $\exists\ \alpha$ which restricts to $\tilde{\alpha}$.
The condition on triple overlaps is vacuous
i.e. gluing <u>exists</u>.

<u>Thm</u>: BG, X/G, $\mathcal{M}_g$, $\mathcal{C}_g$, Vect_n, Coh, Qcoh are all stacks in (étale, smooth, flat, ...) topology. $(g \geq 2)$.
Zariski.

<u>Rem</u>: Non-trivial — Descent theory.

Take $\mathcal{M}_g$: Obj over S are $\pi: C \to S$ — C scheme, S scheme, sm proper curves of genus $g \geq 2$.

Descent for proper morphisms is <u>false</u> in general (we may not be able to glue in the étale topology to get a scheme). It is true for ~~of~~ Qcoh sheaves, hence for affine maps, and also for ~~proper maps~~ "polarized" proper maps i.e. obj ~~$\longleftrightarrow$~~: $(X \xrightarrow{\pi} S$, $\mathcal{L}$ a line bundle on X π-relatively ample$)$.

morphisms:
$$\begin{array}{ccc} X_S & \xrightarrow{g} & X_T \\ \pi\downarrow & \square & \downarrow \\ S & \xrightarrow[f]{} & T \end{array}$$
along with $g^* \mathcal{L}_T \xrightarrow{\sim} \mathcal{L}_S$.

Basically, in this case one considers the proj. coordinate ring $\pi_*(\mathcal{L}^n)$ and descends this algebra. For curves the relative canonical bundle provides a <u>canonical</u> polarization $\Rightarrow$ descent works.

<u>Example of failure of descent</u>: Fund. Alg. Geo. 4.4.2.

<u>Fiber Products</u> — F, G, H CFG's over ~~S~~ $\mathcal{C}$.

$$\begin{array}{ccc} \boxed{F \times_G H} & \to & F \\ \downarrow & & \downarrow f \\ H. & \xrightarrow[h]{} & G \end{array}$$

<u>obj</u> over $A \in \mathcal{C}$ are (x, y, ψ), $x \in F(A)$, $y \in H(A)$

$\psi: f(x) \to h(y)$. an iso.

$$\begin{tikzcd} (x', y', \psi') \arrow[r, bend left, "F"] \arrow[r, bend left=50, "G"] \arrow[d, dotted] & (x, y, \psi) \arrow[d, dotted] \\ A \arrow[r] & B \end{tikzcd}$$
$\leftarrow$ <u>maps</u>

such that $\longleftarrow$

$$\begin{array}{ccc} f(x') & \xrightarrow{\psi'} & g(y') \\ \uparrow & & \uparrow \\ f(x) & \xrightarrow{\psi} & g(y) \end{array}$$

Examples: ①

$$\begin{tikzcd} \square \arrow[r] \arrow[d] & \bullet \arrow[d, "\text{triv.}"] \\ X \arrow[r, "E"] & BG \end{tikzcd}$$

$E \to X$ a G-bundle.

$\square(T)$: $(T \xrightarrow{f} X,\ \psi : T\times G \xrightarrow{\sim} f^*E)$. eqv. to

$(T \xrightarrow{f} X,$ section of $f^*E \to T)$ eqv. to

$(T \xrightarrow{\tilde{f}} E)$.

So $\square \cong E$.

②

$$\begin{tikzcd} \square \arrow[r] \arrow[d] & C_g \arrow[d] \\ S \arrow[r, "C"] & M_g \end{tikzcd}$$

$C \xrightarrow{\pi} S$

$\square(T)$: $\left(T \xrightarrow{f} S,\ \mathcal{C}' \xrightarrow{\sigma} T,\ \begin{tikzcd} C' \arrow[r, "\sim"] \arrow[d] & f^*C \arrow[dl] \\ T & \end{tikzcd}\right)$. eqv.

$(\sigma : T \to f^*C)$.

So $\square \cong C$.

③ Exercise:

$$\begin{tikzcd} \square = E \arrow[r] \arrow[d] & X \arrow[d] \\ T \arrow[r, "\varphi"] & {[X/G]} \end{tikzcd}$$

φ: $\begin{tikzcd} E \arrow[r, "G\text{-eqv.}"] \arrow[d] & X. \\ T & \end{tikzcd}$

Representable Morphisms

Def : $p : F \to G$ is representable if for any scheme X and $f : X \to G$, the fiber product $F \times_G X$ is a scheme

$$
\begin{tikzcd}
\text{Scheme} = \square \arrow[r] \arrow[d] & F \arrow[d] \\
X \arrow[r] & G
\end{tikzcd}
$$

Any property of morphisms of schemes that is stable under base change applies to representable morphisms.

Ex. (i) $\mathcal{C}_g \to \mathcal{M}_g$ smooth proper.

(ii) $\bullet \to BG$

(iii) $X \to [X/G]$

~~DM Stacks~~ : ~~Idea~~ - ~~a stack with~~ ~~an étale surjective,~~

~~map~~

~~DM stack : Idea : $\mathfrak{X}$ is a DM stack if.~~

Looking ahead ; We'll define a DM-stack as a CFG over Schemes that is (a) a stack and (b) admits a surjective étale map from a scheme (an "atlas").

i.e. $\mathfrak{X}$ is étale locally like a scheme.

$$
\begin{array}{c}
U \\
\downarrow \text{ étale surj.} \\
\mathfrak{X}
\end{array}
$$

However, to make sense of this, the map $U \to \mathfrak{X}$ must be representable.

Let us see what this entails:

$$\begin{tikzcd} U \times_{\mathfrak{X}} V \arrow[r] \arrow[d] & U \arrow[d, "\alpha"] \\ V \arrow[r, "\beta"'] & \mathfrak{X} \end{tikzcd}$$

scheme

$\alpha \in \mathfrak{X}(U)$

$\beta \in \mathfrak{X}(V)$

$$U \times_{\mathfrak{X}} V \,(T) = \{ (T \xrightarrow{f} U,\ T \xrightarrow{g} V,\ \psi : f^*\alpha \xrightarrow{\sim} f^*\beta) \}$$

$$=: \underline{\mathrm{Isom}}(\alpha, \beta).$$

<u>Claim</u>: $U \times_{\mathfrak{X}} V \cong \square$ where

$$\begin{tikzcd} \square \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathfrak{X} \arrow[d, "\Delta"] \\ U \times V \arrow[r, "(\alpha,\beta)"'] & \mathfrak{X} \times \mathfrak{X} \end{tikzcd}$$

<u>Pf</u>: $\square(T) = \{ (T \xrightarrow{(f,g)} U \times V,\ T \xrightarrow{\gamma} \mathfrak{X},\ \psi : (f^*\alpha, g^*\beta) \xrightarrow{\sim} (\gamma, \gamma) \}$. eqv. to

$\{ (T \to U,\ T \to V,\ \psi : f^*\alpha \xrightarrow{\sim} g^*\beta \}$.

<u>Def</u>: A stack $\mathfrak{X}$ is <u>Deligne-Mumford</u> <u>algebraic</u> if:

(1) Δ is representable, quasicompact, and separated.

(2) There is a scheme U and étale surj $U \to \mathfrak{X}$. ("atlas").

<u>Recall</u>: For a scheme Δ_F is an embedding.

i.e. $U \times_{\mathfrak{X}} V = \underline{\mathrm{Isom}}(\alpha, \beta)$

$$\begin{tikzcd} U \times_{\mathfrak{X}} V \arrow[d] \\ U \times V \end{tikzcd}$$

$U \xrightarrow{\alpha} \mathfrak{X}$

$V \xrightarrow{\beta} \mathfrak{X}.$

The failure of this being an embedding

$\Leftrightarrow$ presence of nontrivial automorphisms.

Roughly, the conditions on Δ mean that the failure is not too much.

<u>Rem</u>: We know (1) for M_g.