

Deligne-Mumford Stacks:

Oct 23

C a category. We defined a CFG over C as a generalization of the notion of a contravariant functor to Sets.

In particular, $F: C^{op} \rightarrow \underline{\text{Sets}}$ gives

$$\underline{F} = \text{Obj} : (A, \omega) \quad A \in \text{ob } C \quad \omega \in F(A)$$

$$\text{maps} : \quad \omega_A \xleftarrow{F(f)} \omega_B$$

$$A \xrightarrow{f} B \quad \text{in } C$$

In particular, given $X \in \text{ob } C$, we get the functor $\text{Maps}(-, X)$, and thus \underline{X} , a fibered category. Concretely:

$$\underline{X} : \text{Obj} : (A, f: A \rightarrow X) \quad \text{morr} : \quad A \rightarrow B \begin{matrix} \nearrow f \\ \nearrow g \end{matrix} X$$

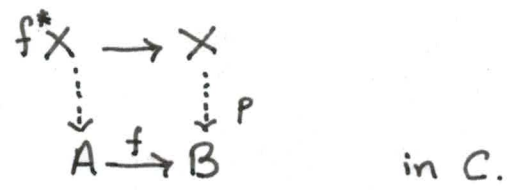
Lemma (Yoneda): Consider a CFG $p: F \rightarrow C$. Then $\text{Hom}(\underline{X}, F)$ is a category (objects = ~~functors~~ maps of CFG's, morphisms = Nat. transf).

Then $\text{Hom}(\underline{X}, F) \xrightarrow{\sim} F(X)$ given by

$$p \longmapsto p(X, \text{id}) \quad \text{is an equivalence.}$$

Definition of a Stack - Recall sheaf = functor + gluing conditions.

Likewise, a stack will be a CFG with gluing conditions. To describe the gluing conditions, it is better to think of a CFG as a "groupoid valued functor" than a category. That is, make choices of pull backs, denoted by upper $\#$.

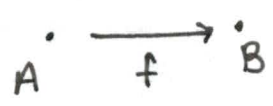


Also, given

$$X \rightarrow Y \in F(B)$$



! $f^*X \rightarrow f^*Y$
denote by $f^*\alpha$.



Then $f^*: F(B) \rightarrow F(A)$ is a functor.

\downarrow \downarrow
 Groupoid groupoid.

Now assume that we have a (Grothendieck) topology on C , i.e. a notion of when a map $U \rightarrow A$ is a covering.

Examples

- ① (surjective), Zariski covers.
- ② étale covers
- ③ smooth covers
- ④ flat (and finite presentation) covers.
- etc. for $C = \underline{\text{schemes}}$.

Def: A CFG $p: F \rightarrow C$ is a stack if the following two conditions hold

① Descent for morphisms: For every covering $U \xrightarrow{\pi} A$ and $\alpha, \beta \in F(A)$ if we are given $\tilde{f}: \pi^* \alpha \rightarrow \pi^* \beta$ such that $\text{pr}_1^*(\tilde{f}) = \text{pr}_2^*(\tilde{f})$ ($U \times_A U \rightrightarrows U$), then $\exists!$ $f: \alpha \rightarrow \beta$ s.t. $\tilde{f} = \pi^* f$

② Descent for objects: For every covering $U \xrightarrow{\pi} A$, if we are given $\tilde{\alpha} \in F(U)$ and $g: \text{pr}_1^* \tilde{\alpha} \rightarrow \text{pr}_2^* \tilde{\alpha}$ such that

$$\text{pr}_{12}^*(g) \circ \text{pr}_{23}^*(g) = \text{pr}_{13}^*(g), \text{ there exists } \alpha \in F(A)$$

such that along with $i: \pi^* \alpha \rightarrow \tilde{\alpha}$ s.t.

$$\begin{array}{ccc}
 \text{pr}_1^* \pi^* \alpha & \xrightarrow{\text{pr}_1^* i} & \text{pr}_1^* \tilde{\alpha} \\
 \parallel & & \downarrow g \\
 \text{pr}_2^* \pi^* \alpha & \xrightarrow{\text{pr}_2^* i} & \text{pr}_2^* \tilde{\alpha}
 \end{array}$$

Example: Suppose $F = \underline{F}$ for a functor $F: C^{\text{op}} \rightarrow \underline{\text{sets}}$

Then $F(A):$ $\begin{array}{c} \bullet \\ \circ \end{array} \xrightarrow{\text{id}} \begin{array}{c} \bullet \\ \circ \end{array} \xrightarrow{\text{id}} \begin{array}{c} \bullet \\ \circ \end{array} \xrightarrow{\text{id}} \begin{array}{c} \bullet \\ \circ \end{array}$ so

①: Given $\tilde{f}: \pi^* \alpha \rightarrow \pi^* \beta \iff \pi^* \alpha = \pi^* \beta$ } conclusion $\alpha = \beta$ i.e. F is a "separated presheaf" (gluing is unique, if exists)

the condition on $U \times_A U$ is vacuous.

②: Given $\tilde{\alpha}$. The existence of $g \iff \text{pr}_1^* \tilde{\alpha} = \text{pr}_2^* \tilde{\alpha}$ } conclusion $\exists \alpha$ which restricts to $\tilde{\alpha}$. i.e. gluing exists.

The condition on triple overlaps is vacuous

Thm: $BG, X/G, M_g, C_g, Vect_n, Coh, Qcoh$ are all stacks in (étale, smooth, flat, ...) topology. ($g \geq 2$).
Zariski.

Rem: Non-trivial — Descent theory.

Take M_g : Obj over S are $\pi: C \rightarrow S$ C scheme, S scheme sm proper curves of genus $g \geq 2$.

Descent for proper morphisms is false in general (we may not be able to glue in the étale topology to get a scheme). It is true for ~~aff~~ $Qcoh$ sheaves, hence for affine maps, and also for ~~proper maps~~ "polarized" proper maps i.e. obj $\Leftrightarrow (X \xrightarrow{\pi} S, L$ a line bundle on X π^* -relatively ample).

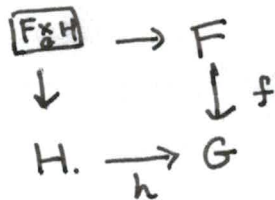
morphisms:
$$\begin{array}{ccc} X_S & \xrightarrow{g} & X_T \\ \pi \downarrow & \square & \downarrow \\ S & \xrightarrow{f} & T \end{array}$$
 along with $g^* L_T \xrightarrow{\sim} L_S$.

Basically, in this case one considers the proj. coordinate ring $\Pi_*(L^n)$ and descends this algebra. For curves the relative canonical bundle provides a canonical polarization \Rightarrow descent works.

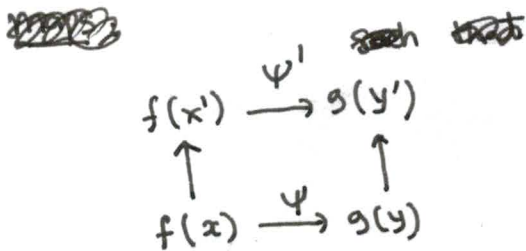
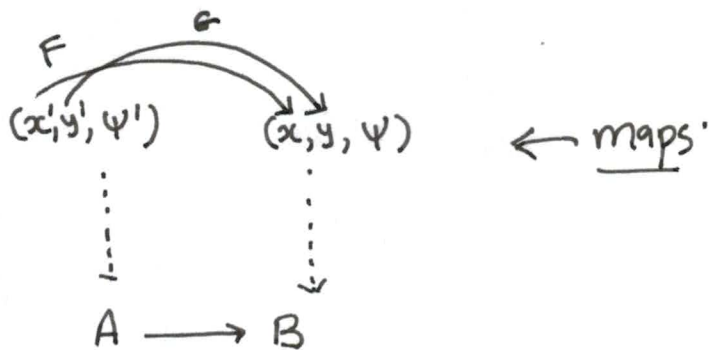
Example of failure of descent: Fund. Alg. Geo. 4.4.2.

Fiber Products - F, G, H CFG's over \bullet, C .

Obj over $A \in C$ are (x, y, Ψ)
 $\begin{array}{cc} \uparrow & \uparrow \\ F(A) & H(A) \end{array}$



$\Psi: f(x) \rightarrow h(y)$. an iso.



such that \leftarrow

Examples: ①



$\square (T)$: $(T \xrightarrow{f} X, \psi: T \times G \xrightarrow{\sim} f^*E)$ eqv. to

$(T \xrightarrow{+} X, \text{section of } f^*E \rightarrow T)$ eqv. to

$(T \xrightarrow{f} E)$.

So $\square \cong E$.

②

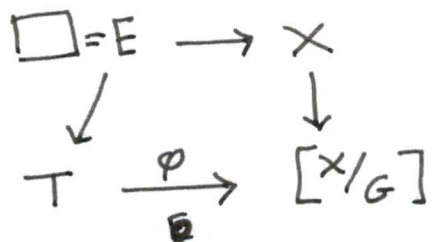


$\square (T)$: $(T \xrightarrow{+} S, \psi' \xrightarrow{\sigma} \bullet T, \begin{array}{c} C' \xrightarrow{\sim} f^*C \\ \downarrow \swarrow \\ \bullet T \end{array})$ eqv.

$(\sigma: T \rightarrow f^*C)$.

So $\square \cong C$.

③ Exercise:



$\varphi: \begin{array}{ccc} E & \xrightarrow{G\text{-equiv.}} & X \\ \downarrow & & \\ T & & \end{array}$

Representable Morphisms

Def: $p: F \rightarrow G$ is representable if for any scheme X and $f: X \rightarrow G$, the fiber product $F \times_G X$ is a scheme

$$\begin{array}{ccc} \text{Scheme} = \square & \longrightarrow & F \\ \downarrow & & \downarrow \\ X & \longrightarrow & G \end{array}$$

Any property of morphisms of schemes that is stable under base change applies to representable morphisms.

- Ex.
- (i) $C_g \rightarrow M_g$ smooth proper.
 - (ii) $\bullet \rightarrow BG$
 - (iii) $X \rightarrow [X/G]$

~~DM stack~~: ~~idea~~ - a stack ~~with~~ ~~a~~ ~~an~~ étale surjective map

~~DM stack~~ ~~idea~~: ~~\mathcal{X} is a DM stack if~~

Looking ahead: We'll define a DM-stack as a CFG over schemes that is (a) a stack and (b) admits a surjective étale map \mathcal{U} from a scheme (an "atlas").

i.e. \mathcal{X} is étale locally like a scheme.

However, to make sense of this, the map

$U \rightarrow \mathcal{X}$ must be representable.

$$\begin{array}{ccc} U & & \\ \downarrow & \text{étale surj.} & \\ \mathcal{X} & & \end{array}$$

Let us see what this entails:

$$\begin{array}{ccc}
 U \times_{\mathcal{X}} V & \longrightarrow & U \quad \text{scheme} \\
 \downarrow & & \downarrow \alpha \\
 V & \xrightarrow{\beta} & \mathcal{X}
 \end{array}
 \quad \begin{array}{l}
 \alpha \in \mathcal{X}(U) \\
 \beta \in \mathcal{X}(V)
 \end{array}$$

$$\begin{aligned}
 U \times_{\mathcal{X}} V(\tau) &= \{ (\tau \xrightarrow{f} U, \tau \xrightarrow{g} V, \psi: f^* \alpha \xrightarrow{\sim} g^* \beta) \} \\
 &\equiv: \underline{\text{Isom}}(\alpha, \beta).
 \end{aligned}$$

Claim: $U \times_{\mathcal{X}} V \cong \square$ where

$$\begin{array}{ccc}
 \square & \longrightarrow & \mathcal{X} \\
 \downarrow & \square & \downarrow \Delta \\
 U \times V & \xrightarrow{(\alpha, \beta)} & \mathcal{X} \times \mathcal{X}
 \end{array}$$

Pf: $\square(\tau) = \{ (\tau \xrightarrow{(f,g)} U \times V, \tau \xrightarrow{\gamma} \mathcal{X}, \psi: (f^* \alpha, g^* \beta) \xrightarrow{\sim} (\gamma, \gamma) \}$. eqv. to

$$\{ (\tau \xrightarrow{f} U, \tau \xrightarrow{g} V, \psi: f^* \alpha \xrightarrow{\sim} g^* \beta) \}.$$

Def: A stack \mathcal{X} is Deligne-Mumford algebraic if:

- (1) Δ is representable, quasicompact, and separated.
- (2) There is a scheme U and étale surj $U \rightarrow \mathcal{X}$. ("atlas").

Recall: For a scheme Δ_F is an embedding.

i.e.

$$\begin{array}{ccc}
 U \times_{\mathcal{X}} V = \underline{\text{Isom}}(\alpha, \beta) & & U \xrightarrow{\alpha} \mathcal{X} \\
 \downarrow & \swarrow & V \xrightarrow{\beta} \mathcal{X} \\
 U \times V & &
 \end{array}$$

The failure of this being an embedding \Leftrightarrow presence of nontrivial automorphisms.

Roughly, the conditions on Δ mean that the failure is not too much.

Rem: We know (1) for M_g .