~~Deligne-Mumford~~ Moduli of Curves – Oct 28.

Algebraic Stack: Roughly speaking, an algebraic stack is a stack that "locally" looks like a scheme (or equivalently, the spectrum of a ring). i.e. we must have an "atlas" $U \xrightarrow{\pi} \mathfrak{X}$ for $\mathfrak{X}$, where π is a covering in the appropriate sense., and U is a scheme.

① π étale $\longleftrightarrow$ Deligne-Mumford stack

② π smooth $\longleftrightarrow$ Artin stack.

However, to make sense of such properties for π, it must be representable. Let us see what this entails.

$$\begin{tikzcd} U \times_{\mathfrak{X}} V \arrow[r] \arrow[d] & U \arrow[d, "\alpha"] \\ V \arrow[r, "\beta"'] & \mathfrak{X} \end{tikzcd}$$

scheme

$\alpha \in \mathfrak{X}(U)$
$\beta \in \mathfrak{X}(V)$.

$$U \times_{\mathfrak{X}} V\,(T) = \{ (f: T \to U,\ g: T \to V,\ \psi: f^*\alpha \xrightarrow{\sim} g^*\beta \}$$
$$=: \mathrm{Isom}(\alpha, \beta).$$

Rem: $\mathfrak{X}$ stack $\Rightarrow$ $\mathrm{Isom}(\alpha,\beta)$ is a sheaf (the first condition in the def.)

Prop: We have $U \times_{\mathfrak{X}} V = (U \times V) \times_{\mathfrak{X} \times \mathfrak{X}}^{\Delta} \mathfrak{X}$

Pf:

$$\begin{tikzcd} \square \arrow[r] \arrow[d] & U \times V \arrow[d] & (\alpha,\beta) \\ \mathfrak{X} \arrow[r, "\Delta"] & \mathfrak{X} \times \mathfrak{X} & \end{tikzcd}$$

$\Delta: \mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$
$\gamma \mapsto (\gamma, \gamma)$

$$\square(T) = \{ (f: T \to U,\ g: T \to V,\ \gamma \in \mathfrak{X}(T),\ (f^*\alpha, g^*\beta) \xrightarrow{\sim} (\gamma,\gamma) \}$$
$$\Big\downarrow \text{eqv.}$$
$$\{ (f, g,\ f^*\alpha \xrightarrow{\sim} g^*\beta \} = U \times_{\mathfrak{X}} V\,(T).$$

$\square$.

So, $U \times_{\mathfrak{X}} V$ will be a scheme if Δ is representable. In fact the converse is also true.

Def: A Deligne-Mumford stack is a stack $\mathcal{X}$ such that

(1) $\Delta : \mathcal{X} \to \mathcal{X}\times\mathcal{X}$ is representable, separated, quasi-compact

(2) There is a scheme U and an étale surjective morphism
$$U \longrightarrow \mathcal{X}$$
(called an "atlas").

Rem:
DM $\longleftrightarrow$ étale atlas.
Artin $\longleftrightarrow$ smooth atlas.

~~Rem: For a DM stack $\mathcal{X}$~~

Rem on the diagonal:
$$\begin{tikzcd}
 & & \mathcal{X} \arrow[r, "\Delta"] & \mathcal{X}\times\mathcal{X} \\
\text{either } \phi \text{ or } \mathrm{Aut}(\alpha) & & \mathrm{Isom}(\alpha,\beta) \arrow[ll] \arrow[r] \arrow[u] & \mathrm{Spec}\,\mathbb{C}. \arrow[u, "{(\alpha,\beta)}"']
\end{tikzcd}$$

For a scheme, Δ is a (locally closed) embedding.

Prop: Let $\mathcal{X}$ be a DM stack. Then Δ is unramified.
~~the~~ (In particular, for any $\alpha \in \mathcal{X}(k)$ the group $\mathrm{Aut}(\alpha)$ is finite. In fact, the aut scheme is finite and ~~non~~ reduced.)

Pf:
$$\begin{tikzcd}
V_2 \arrow[rrr] \arrow[dr] \arrow[ddd, "\text{surj. étale}"'] & & & V_3 \arrow[dl] \arrow[ddd, "\text{surj. étale}"] \\
& U \arrow[r, hook] \arrow[d] & U\times U \arrow[d] & \\
& \mathcal{X} \arrow[r, "\Delta"] & \mathcal{X}\times\mathcal{X} & \\
V_1 \arrow[ur] \arrow[rrr] & & & V \arrow[ul]
\end{tikzcd}$$

unramified $\Leftrightarrow$ "immersion"
$$\begin{tikzcd}
x\in U \arrow[r, "\varphi"] & V \\
U' \arrow[r, hook] \arrow[u, "\text{étale}"] & V' \arrow[u, "\text{étale}"]
\end{tikzcd}$$
unram at x $\Updownarrow$ $\exists$ étale neighborhoods of x, $\varphi(x)$ s.t. φ is an embedding on these neighborhoods

$\square$.

For M_g, we know (1).
We also know that Δ is unramified.

Prop: / **Thm:** Let $\mathcal{X}$ be a stack over a Noetherian base scheme S such that

(1) Δ is repr., q.c., separated, and unramified.

(2) $\exists$ U of finite type over S and a <u>smooth</u> surj

$$\pi : U \to \mathcal{X}$$

Then $\mathcal{X}$ is D.M.

<u>i.e.</u> Given that Δ is unramified, a smooth atlas $\Rightarrow$ an étale atlas.

Cor: M_g is a D.M. stack.

<u>Pf</u>: We only need to produce a smooth atlas.

Let $d > 2g-2$. Consider $H \subset$ <u>Hilb</u> the open set parametrizing non deg. smooth curves of arithmetic genus g and degree d in $\mathbb{P}^r$, where $r = d-g+1$.

<u>Claim</u>: $H \to M_g$ is smooth.

<u>Pf</u>: Suffices to check the infinitesimal lifting criterion.

$$\begin{tikzcd} \mathcal{C} \arrow[r, hook] \arrow[d] & \mathcal{C}' \arrow[d] \\ \mathrm{Spec} A \arrow[r, hook] & \mathrm{Spec} A' \end{tikzcd}$$

<u>Given</u>: An embedding $\mathcal{C} \subset \mathbb{P}^r_A$ of deg d

<u>Want</u>: An extension $\mathcal{C}' \subset \mathbb{P}^r_{A'}$.

Let $L_A = \mathcal{O}(1)$ of $\mathbb{P}^r_A$ restricted to $\mathcal{C}$. We have an iso.

$$A^{r+1} \xrightarrow{\sim} H^0(\mathcal{C}, L_A). \quad \text{—— } ①$$

Extend L_A to a line bundle $L_{A'}$ on $\mathcal{C}'$.

Then $H^0(\mathcal{C}', L_{A'})$ is locally free of rank $(r+1)$ by coh & base change.

So we get $H^0(\mathcal{C}', L_{A'}) \xrightarrow{\sim} A'^{r+1}$ extending ①.

Thus $\mathcal{C}' \to \mathbb{P}^r_{A'}$. (embedding automatic). $\square$

<u>Pf of thm</u> <u>(Sketch)</u>: In char 0, or say over $\mathbb{C}$.

Take a point $x : \operatorname{Spec}\mathbb{C} \to \mathfrak{X}$. We want to produce an étale chart for $\mathfrak{X}$ around x. We have:

$$\begin{tikzcd} U_x \arrow[r, "i"] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d, "\text{smooth.}"] \\ x \arrow[r] & \mathfrak{X} \end{tikzcd}$$

Note that:

$$\begin{tikzcd} U_x \arrow[r, "i"] \arrow[d] \arrow[dr, phantom, "\square."] & x \times U \arrow[d] \\ x \arrow[r, "\Delta"] & \mathfrak{X} \times \mathfrak{X} \end{tikzcd}$$

$\Rightarrow$ $i : U_x \to U$ is unramified $\Rightarrow$

$\exists$

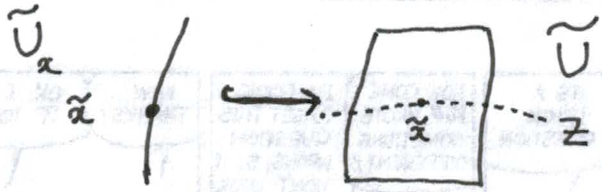
$$\begin{tikzcd} \tilde{U}_x \arrow[r, hook, "i"] \arrow[d, "\text{étale}"'] \arrow[dr, phantom, "\square"] & \tilde{U} \arrow[d, "\text{étale}"] \\ U_x \arrow[r, "i"] & U \end{tikzcd}$$

i is an embedding

Pick $\tilde{x} \in \tilde{U}_x$.

Since $U \to \mathfrak{X}$ is smooth, $\tilde{U}_x \to \operatorname{Spec}\mathbb{C}$ is smooth.

$\Rightarrow$ $\tilde{x} \in \tilde{U}_x$ is cut out by a regular sequence $t_1, \dots, t_n$.



Lift t_i to $\tilde{t}_i$ on $\tilde{U}$ and set $Z = V(\tilde{t}_i) \subset \tilde{U}$.

<u>Claim</u>: The map $Z \to \mathfrak{X}$ is étale over x.

<u>Pf</u>: We need to check that the map σ is étale over $\tilde{x}$.

$$\begin{tikzcd} \tilde{U} \times_{\mathfrak{X}} Z \arrow[r] \arrow[d, "\sigma"] \arrow[dr, phantom, "\square"] & Z \arrow[d] \\ \tilde{U} \arrow[r] & \mathfrak{X} \end{tikzcd}$$

Now $\tilde{U} \times_{\mathfrak{X}} \tilde{U} \to \tilde{U}$ is smooth and $\tilde{U} \times_{\mathfrak{X}} Z \hookrightarrow \tilde{U} \times_{\mathfrak{X}} \tilde{U}$ is defined by the vanishing of $\tilde{t}_1, \dots, \tilde{t}_n$.

Furthermore, over $\tilde{x} \in \tilde{U}$ we have

$$\begin{tikzcd} \tilde{x} \arrow[rr] \arrow[d] \arrow[drr, phantom, "\square."] & & \tilde{U} \times_{\mathcal{X}} Z \arrow[d] \\ \tilde{U}_x = \tilde{x} \times_{\mathcal{X}} \tilde{U} \arrow[rr] & & \tilde{U} \times_{\mathcal{X}} \tilde{U} \end{tikzcd}$$

def. by $t_1, \dots, t_n$. $\leftarrow$

$\Rightarrow$ by the Jacobian Criterion that $\tilde{U} \times_x Z \to \tilde{U}$ is smooth of rel. dim 0 (i.e. étale) in a neighborhood of $\tilde{x} \in \tilde{U}$.

Thus $\tilde{U} \to \mathcal{X}$ is étale over x.

$\square$.

<u>Examples</u>: (1) G étale group scheme over S

$\Rightarrow$ BG is a DM stack.

(2) G smooth affine groupscheme $/S$ acting on X.

such that the stabilizers of geometric points of X are ~~zero-dim~~ finite and reduced. (automatic in char 0).

Then $[X/G]$ is a DM stack.

$$\begin{tikzcd} \mathrm{Stab}_x \arrow[r] \arrow[d] \arrow[dr, phantom, "\square."] & X_\Delta \arrow[r] \arrow[d] & {[X/G]} \arrow[d] \\ (x,x) \arrow[r] & X \times X \arrow[r] & {[X/G] \overset{\Delta}{\times} [X/G]} \end{tikzcd}$$

$$\begin{tikzcd} X_\Delta = \square \arrow[r] \arrow[d] & G \times X \arrow[d, "a \mid p"] \\ X \arrow[r, "\Delta"] & X \times X \end{tikzcd}$$