Moduli of Curves - Oct 30

Last time - unramified diagonal + smooth atlas $\Rightarrow$ étale atlas

Smooth atlas for M_g: Fix $d > 2g-2$.

H = Open subset of Hilb param. smooth nondeg. $C \subset \mathbb{P}^N$ of degree d, genus g.

$H \to M_g$ smooth.

A better presentation - Let $k \geq 2$ and $d = (2g-2)\cdot k$.

Then there exists a closed ~~subset~~ $H_k \subset H$ that parametrizes C where $\mathcal{O}_C(1) \cong \omega_C^k$.

How to construct H_k:
$$\begin{array}{c} C \subset \mathbb{P}^N_H \\ \downarrow \pi \\ H \end{array} \qquad L = \mathcal{O}_C(1) \otimes \omega_C^{-k}.$$

Claim: $\exists$ closed subscheme $H_k \subset H$ s.t. $\varphi: S \to H$ factors through H_k iff $\pi_*(L)$ is locally free of rank 1 on S.

Pf: Exercise using cohomology and base change.

Functorially: $H_k(S) = \left\{ \begin{array}{c} C \subset \mathbb{P}^N \\ \downarrow \pi \\ S \end{array} \middle| \begin{array}{l} \pi\text{ - smooth, non-deg fibers} \\ \pi_*(\mathcal{O}_{\mathbb{P}^N}(1) \otimes \omega_C^{-k}) \text{ locally free of rank 1} \end{array} \right\}.$

$\cong \left\{ \begin{array}{c} C \\ \downarrow \pi \\ S \end{array} \middle| \text{ isom. } \mathbb{P}(\pi_* \omega_C^k) \cong \mathbb{P}^N_S \right\}.$

Prop.: $M_g \cong [H_k / PGL_N]$.

Pf: Skip.

Properties of Stacks (algebraic stacks): Let $\mathcal{X}$ be a DM / Artin stack over S. Suppose P is any property that is local in the smooth topology. Then we say that $\mathcal{X}/S$ has P iff any smooth atlas $U \to \mathcal{X}$ has P. Similarly for étale.

Ex. M_g is smooth ~~/k~~. /k.

M_g is equidimensional of dim $3g-3$.

~~$M_g/\mathbb{C}$~~

Presentations: Let $\mathfrak{X}$ be ~~an atlas~~ an algebraic stack and $\pi: U \to \mathfrak{X}$ an atlas. [Analogy – X a scheme and $\coprod U_i \to X$ a covering).

Let $R = U \times_{\mathfrak{X}} U$.

$$\begin{array}{ccc} R & \to & U \\ \downarrow & & \downarrow \\ U & \to & \mathfrak{X} \end{array}$$

$R = \coprod U_i \cap U_j$

$R \overset{i}{\underset{j}{\rightrightarrows}} U$

We have two maps

$$R \overset{s}{\underset{t}{\rightrightarrows}} U.$$

Furthermore, we have a "composition" map

$$c: R \times_{t, U, s} R \to R$$

$$c: (U \times_{\mathfrak{X}} U) \times_{2, \mathfrak{X}, 1} (U \times_{\mathfrak{X}} U) \to U \times_{\mathfrak{X}} U.$$

$$\| $$

$$\mathrm{Isom}(\alpha, \alpha) \times_U \mathrm{Isom}(\alpha, \alpha) \to \mathrm{Isom}(\alpha, \alpha).$$

$$((u, v), \psi), (v, w), \varphi \to (\psi \circ \varphi).$$

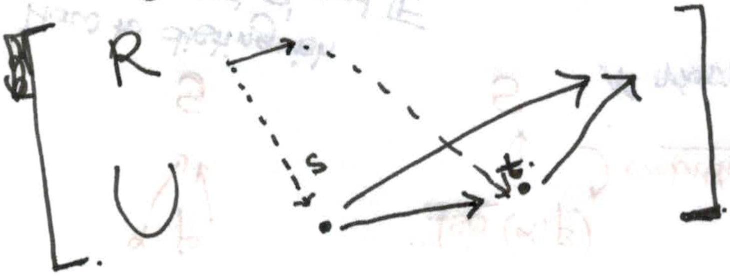
$U_i \cap U_j \cap U_j \cap U_k \to U_i \cap U_k$

and an inverse map

$$R \to R.$$

and an identity map $e: R \to R$.

The gadget $(R \rightrightarrows U, s, t, e, i)$ forms a Groupoid.



The stack $\mathfrak{X}$ is "determined" by this groupoid.

~~Conversely any (sufficiently nice)~~

Sheaves on Stacks:

Sheaves on schemes - $X \rightsquigarrow$ Category $\underline{\text{Schemes}}_X = \{U, f: U \to X\}$.

~~Category~~ ~~Presh~~

presheaf = functor $\underline{\text{Schemes}}_X^{op} \to$ Sets.

sheaf = presheaf + gluing on coverings.

We have all this machinery for stacks by construction. -

$\mathcal{X} \rightsquigarrow$ itself a category. obj. $= \{(U, \alpha: U \to \mathcal{X})\}$.

presheaf = contr. var. functor.

sheaf = presheaf + gluing.

Ex. $\mathcal{O}_{\mathcal{X}}$. $U \mapsto \Gamma(U, \mathcal{O}_U)$. = Structure sheaf.

$\rightsquigarrow$ $\mathcal{O}_{\mathcal{X}}$-modules, quasicoherent, coherent, locally free ....

Eqv. way : The data of a quasi-coherent sheaf $\mathcal{F}$ on $\mathcal{X}$ is.

① The data of a quasi-coh. sheaf F_α on every ~~[illegible]~~.

② For a map $f: U \xrightarrow{s} V$ in $\mathcal{X}$, a choice of iso. of $s^*: F_U \to$ ...

eqv: if $R \overset{s}{\underset{t}{\rightrightarrows}} U$ is a groupoid presentation of $\mathcal{X}$ then.

① The data of a sheaf F on U

② iso $s^*F \to t^*F$

s.t. ~~on~~ this behaves nicely with the composition map.