Deformations and Obstructions

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Recall that we were trying to see whether the infinitesimal lifting criterion holds for Hilb_X.

k = alg closed field. X/k a projective scheme.
A = local k-algebra. (Artin).
$\tilde{A} \to A$ a small extension $\left(0 \to k \xrightarrow{\varepsilon} \tilde{A} \to A \to 0 \right)$.

$$\begin{array}{ccc} Z_A & \hookrightarrow & X_A \\ \cap & & \cap \\ ?? & \hookrightarrow & X_{\tilde{A}} \end{array}$$

Extend locally and glue.
Local problem: R a k-algebra of finite type

$$\begin{array}{ccc} I_A & \subset & R_A \\ \uparrow & & \uparrow \\ ?? & \subset & R_{\tilde{A}} \end{array}$$

$I_A \subset R_A$ an ideal. Want $I_{\tilde{A}}$.

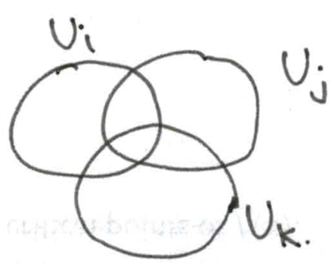
<u>Prop</u>: The set of ideals $I_{\tilde{A}}$ of $R_{\tilde{A}}$ with $\tilde{A}$-flat quotient that lift I_A is either empty or a principal homogeneous space under the action of $\mathrm{Hom}_R(I, R/I) = \mathrm{Hom}_{R/I}(I/I^2, R/I)$.

<u>Remk</u>: Note that the structure group $\mathrm{Hom}_{R/I}(I/I^2, R/I)$ depends only on the central fiber $I \subset R$ and <u>not</u> on the deformation $I_A \subset R_A$.

<u>Def</u>: We say that $Z_A \subset X_A$ is <u>locally unobstructed</u> if there exists an open cover of X such that $Z_A|_U \subset X_A|_U$ extends to an $\tilde{A}$-flat ~~deformation~~ $\cancel{Z_{\tilde{A}}|_U \subset X_{\tilde{A}}|_U}$. deformation for every U in the cover.

Assume this is the case:

X:



On each U_i pick a deformation $Z^i \subset X_{\tilde{A}}|_{U_i}$.

To be able to glue, these objects (i.e. their defining ideals) must agree on the overlaps.

On $U_i \cap U_j$ we have $[Z^i]|_{U_{ij}}$ and $[Z^j]|_{U_{ij}}$.

Since deformations form a PHS, we can compare them:

$$\delta_{ij} = [Z^i]|_{U_{ij}} - [Z^j]|_{U_{ij}} \in \mathrm{Hom}_X(\mathcal{I}, \mathcal{O}_Z)|_{U_{ij}} = H^0(U_{ij}, N_{Z/X}).$$

~~The gluing cond~~

For gluing, we want $\delta_{ij} = 0$ for all i,j.

If this is not the case, we can adjust the $[Z^i]$ on U^i to

$[Z^i] + d_i$ where $d_i \in H^0(U_i, N_{Z/X})$.

Then δ_{ij} changes to $\delta_{ij} + d_i - d_j$.

Furthermore (δ_{ij}) satisfies the cocycle condition:

$$\delta_{ij} + \delta_{jk} = \delta_{ik}.$$

Thus the δ_{ij} define a Čech 1-cocycle of $N_{Z/X}$ on $\{U_i\}$, and a global deformation exists if and only if this is a coboundary.

<u>Prop</u>: ~~To a deforma~~ In the setup above, we can associate to $Z_A \subset X_A$ an element $\mathrm{obs} \in H^1(Z, N_{Z/X})$ such that an extension ~~of~~ $Z_{\tilde{A}} \subset X_{\tilde{A}}$ exists if and only if $\mathrm{obs} = 0$.

<u>Summary</u>

Thus, assuming that $Z_A \subset X_A$ is locally unobstructed, we can quantify the global obstruction as an element of $H^1(Z, N_{Z/X})$.

Important Special Case: Local Complete intersections.

It turns out that if $Z \subset X$ is a local complete intersection (that is, if the ideal of Z is generated locally by a regular sequence), then all deformations of $Z \subset X$ are locally unobstructed!

To prove this, we need a small lemma.

Lemma: Let $I \subset R$ be generated by a regular sequence and let $I_A \subset R_A$ be any lift of I_A. Then I_A is also generated by a regular sequence and R_A / I_A is A-flat.

Pf: (Note: My proof in class, although correct in spirit, was wrong in a detail. The claim "a complex that ~~rem~~ is exact mod. max. ideal must already be exact" is clearly false without additional hypotheses. Here is a (hopefully) correct proof.)

Let $f_1, \dots, f_n$ be a reg. seq that generates I and let $f_1^A, \dots, f_n^A$ be lifts of these to I_A. By Nakayama's lemma, they generate I_A.

Consider $R_A \xrightarrow{f_1^A} R_A \to R_A / f_1^A \to 0$. we show that R_A / f_1^A is A-flat & this seq. is exact. To do so, consider

$$\begin{array}{ccccccccc} 0 \to & m_A \otimes_A R_A & \to & R_A & \to & R & \to 0 \\ & \downarrow & & \downarrow f_1^A & & \downarrow f_1 & \\ 0 \to & m_A \otimes_A R_A & \to & R_A & \to & R & \to 0 \end{array}$$

The rows are exact because R_A is A-flat. Chasing the snake, we get

$$0 \to m_A \otimes_A (R_A / f_1^A) \to R_A / f_1^A \to R / f_1 \to 0.$$

This shows (by loc. crit. of flatness) that R_A / f_1^A is A-flat. From

$$0 \to (f_1^A) \to R_A \to R_A / f_1^A \to 0$$

we conclude that (f_1^A) is also A-flat. From

$$0 \to K \to R_A \xrightarrow{f_1^A} (f_1^A) \to 0 \quad : \quad \otimes_A k$$

we get $K \otimes_A k = 0$. By Nakayama, $K = 0$ so $R_A \xrightarrow{f_1^A} R_A$ is injective.

We now continue with R_A replaced by R_A / f_1^A and f_2^A .... $\square$

Prop: Let $I_A \subset R_A$ be a lift of $I \subset R$, where I is gen. by a reg. seq. and $\tilde{A} \to A$ a small ext^n. Then I_A extends to an ideal $I_{\tilde{A}} \subset R_{\tilde{A}}$ with $\tilde{A}$-flat quotient.

Pf: Lift the regular gen. of I_A to $R_{\tilde{A}}$ and set $I_{\tilde{A}}$ to be the ideal they generate. By the lemma applied to $\tilde{A}$, the quotient is automatically flat. $\square$.

Cor: If $Z \subset X$ is a loc. compl. int, then any def. $Z_A \subset X_A$ (flat over A) ~~extends to~~ ~~$Z_{\tilde{A}} \subset X_{\tilde{A}}$~~ is locally unobstructed. (~~and~~ for any $\tilde{A} \to A$). In particular, the only obstruction to lifting $Z_A \subset X_A$ is the global one, lying in $H^1(Z, N_{Z/X})$.

This applies, most importantly, when both Z and X are smooth.

Applications Hilb is smooth at

(1) Rat normal curves in $\mathbb{P}^n$

(2) Smooth curves embeded by a line bundle of deg $> 2g-2$

(3) Canonically embedded smooth curves.

Note on Nakayama's lemma: As some of you observed, we are applying Nak. ~~to non finitely gen. modules~~. ~~Here is the precise statement~~ lemma ~~to~~ in a funny situation. We have $A \to R_A$ and M a fin gen. module over R_A. Let $m \subset A$ be the maximal ideal. Suppose $M \otimes_A A/m = 0$, then we can deduce that $M \otimes_{R_A} R_A/m' = 0$ for all maximal ideals m' of R_A that contain m. In particular, if all max ideals of R_A contain m, then $M \otimes_{R_A} R_A/m' = 0$ $\forall$ max id $m' \subset R_A$ so $M = 0$ by the usual Nakayama.

In our case m is a nilpotent ideal, so all max id. of R_A automatically contain it.