Moduli of Curves : Sept 23

$Z \subset X$, where X is a projective scheme over k.

$H^0(Z, N_{Z/X}) = T_{Z\subset X}\text{Hilb} =$ Space of first order deformations

Suppose $Z \subset X$ is LCI.

$H^1(Z, N_{Z/X}) =$ Space of obstructions.

Hilb is smooth at $[Z\subset X]$ if $H^1(Z, N_{Z/X}) = 0$.

Caution : The converse is not true.

Example : Curves on a K3 surface. / $\mathbb{C}$.

$X \subset \mathbb{P}^3$ a general quartic $\Rightarrow \mathrm{Pic}(X) \cong \mathbb{Z}.\langle H\rangle$.

Consider the Hilbert scheme of curves on X. The Hilb poly is fixed by the class of the curve. Say the class is dH.

Then $\text{Hilb}_X^{dH} \cong \mathbb{P}H^0(X, \mathcal{O}_X(d)) \leftarrow$ smooth.

But $H^1(C, N_{C/X}) = H^1(C, \mathcal{O}_C(C)) = H^1(C, K_C) = \mathbb{C} \neq 0$.

Refinement : Kollar (Thm 2.8 ...)

The local ring of Hilb at $[Z\subset X]$ is a quotient of a regular local k-algebra of dim $h^0(N_{Z/X})$ and the ideal is generated by $h^1(N_{Z/X})$ elts.

Cor : $h^0 - h^1(N_{Z/X}) \leq \dim_Z \text{Hilb}_X \leq h^0(N_{Z/X})$

$\quad\quad \|$

$\chi(N_{Z/X})$ for the case of curves. Z.

Pf - Skip.

This concludes our study of the local properties of Hilb.

Exercise : Show that if $X \subset \mathbb{P}^3$ is a smooth surface of degree ≥ 3, then X has finitely many lines on it.

Applications

① Space of Maps:

Let X, Y be projective schemes. Define the functor.

$$\mathrm{Maps}(X,Y) : T \mapsto \{ f : X \times T \to Y \times T \ \text{ over } T \}.$$

<u>Thm</u>: $\mathrm{Maps}(X,Y)$ is represented by an open subscheme of $\mathrm{Hilb}_{X \times Y}$.

<u>Pf</u>: Consider the natural transformation

$$\mathrm{Maps}(X,Y) \longrightarrow \mathrm{Hilb}_{X \times Y}$$
$$f \longmapsto \Gamma_f = \text{graph of } f \subset (X \times Y)_T.$$

~~Then~~ ~~Maps~~ This identifies $\mathrm{Maps}(X,Y)$ with the subfunctor of $\mathrm{Hilb}_{X \times Y}$ given by

$$T \mapsto \{ Z \subset (X \times Y)_T \mid \pi_1 : Z \to X_T \text{ is an iso.} \}.$$

We show that this is an open subfunctor of $\mathrm{Hilb}_{X \times Y}$.
That is, given $Z \subset (X \times Y)_T$ flat over T, we want to show that the locus of $t \in T$ such that $\pi_1 : Z_t \to X_t$ is an iso. is an open subset of T.

Let $t \in T$ be such a point.

First, there's an open set around t s.t. $\pi_1 : Z_t \to X_t$ has finite fibers (by semicontinuity of fiber dim). Note that π_1 is also proper.
$\Rightarrow$ In a neighborhood of t, $\pi_1 : Z_t \to X_t$ is finite.

So it suffices to check that

$$\mathcal{O}_X \to \pi_* \mathcal{O}_Z$$

is an isomorphism. Note that both are flat sheaves over T and this map is an iso. at t. $\Rightarrow$ (by Nakayama) that it is an iso ~~in~~ an open set containing t

□

Caution: Maps(X,Y) may have infinitely many components.

(But ~~only~~ ~~finite~~ it is quasi-proj if we fix an ample line bundle on $X\times Y$ and fix a Hilb poly of the graph.)

~~Application: Let X,Y be smooth projective curves. $k=\bar{k}$.~~
~~Then~~

Isom:

Let X be a projective scheme. Consider the functor

$$\underline{\mathrm{Isom}}_X : T \mapsto \{ f: X_T \to X_T \text{ iso. } /T \}.$$

Thm: $\underline{\mathrm{Isom}}_X$ is represented by an open subscheme of Maps(X,X).

Pf: Clear.

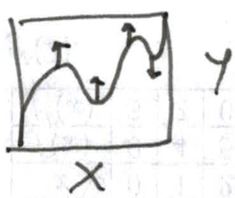
Applications:

① Let X, Y be smooth projective curves over an alg. closed field of char 0. Suppose $g(Y)>1$.

Then there are finitely many nonconstant maps $f: X\to Y$.

Pf: Let us show that Maps(X,Y) is zero dimensional at a nonconstant map $f: X\to Y$.

$$T_f \,\mathrm{Maps}(X,Y) = T_{\Gamma_f} \mathrm{Hilb}_{X\times Y} = H^0(\Gamma_f, N_{\Gamma_f/X\times Y})$$
$$\| $$
$$H^0(X, f^*\underbrace{T_Y}_{\text{negative degree}}) = 0.$$



Now, let us show that Maps(X,Y) is quasi-proj.

Enough to show that there are only finitely many possible Hilb poly.

Fix an ample line bundle L_X on X and L_Y on Y so we get an ample line bundle $L_X \boxtimes L_Y$ on $X \times Y$.

Now $\cancel{\chi(\Gamma}$

$$\begin{aligned} \chi(\Gamma_f, L_X \boxtimes L_Y) &= \chi(X, L_X \otimes f^* L_Y). \\ &= \deg(L_X \otimes f^* L_Y) + g_X - 1 \\ &= \deg(L_X) + (\deg f)(\deg L_Y) + g - 1 \end{aligned}$$

$\Rightarrow$ Enough to show $(\deg f)$ is bounded.

But by Riemann-Hurwitz.

$$g_Y - 2 = (\deg f)(g_X - 2) + \text{Ram}.$$

$\Rightarrow$ $\deg f$ is bounded.

$\triangle$: Does not work in char p.

<u>However</u> : $\cancel{\exists}$ There are finitely many non const. maps of <u>bounded</u> degree.

In particular <u>Isom</u>$_X$ is finite (and reduced!)

<u>Thm</u> : Let X be a smooth proj curve of genus ≥ 2 over an alg closed k. Then <u>Isom</u>$_X$ is finite and reduced.

$\triangle$ <u>Isom</u>$_X$ is NOT always quasi-projective.

<u>Ex.</u> $\mathbb{P}^2$ blown up at 9 points of the intersection of 2 cubics.