Construction of Hilb

Let $X \subset \mathbb{P}^n$ be a projective scheme over k. Let P be a poly.

Thm: $\underline{\mathrm{Hilb}}^P_X$ is represented by a projective scheme $/k$.

Generally: Let $\mathfrak{X} \subset \mathbb{P}^n \times T$ be a flat family of projective schemes over T.

$$\underline{\mathrm{Hilb}}^P_{\mathfrak{X}} : S/T \longmapsto \{ Z \subset \mathfrak{X}_S \text{ flat over } S \text{ with Hilb poly } P \}.$$

Thm: $\underline{\mathrm{Hilb}}^P_{\mathfrak{X}}$ is represented by a projective T-scheme.

Outline of the proof: $(X = \mathbb{P}^n_k)$.

- Exhibit $\underline{\mathrm{Hilb}}^P_X$ as a locally closed subscheme of a grassmannian.

Map $\underline{\mathrm{Hilb}}^P_X \to \mathrm{Gr}$. Fix. m (to be chosen later).

On points:

$$(*) \quad [Z \subset X] \longmapsto [H^0(I_Z(m)) \subset H^0(\mathcal{O}(m))]$$

subspace of rank ..... $\qquad H^0(\mathcal{O}(m)) = \mathrm{Sym}^m(k^{n+1})$.

→ m should be such that $H^0(I_Z(m))$ has fixed rank.

Hilb Poly of I_Z is fixed, say Q. $\left(P + Q = \binom{n+d}{d}\right)$.

So, $h^0(I_Z(m)) = Q(m)$ if $h^i(I_Z(m)) = 0$ for $i > 0$.

So we must choose $m \gg 0$ so that this happens. —— ①

Next, if this map $(*)$ were to be an embedding, it must be injective on points.

i.e. we must be able to recover I_Z from $H^0(I_Z(m))$. — ②

The first technical step in the proof is the existence of an m such that ① and ② hold.

<u>Lemma</u> <u>(Uniform m-lemma)</u>: There exists m (depending on n, P) such that for any ~~~~ ~~and~~ $Z \subset \mathbb{P}^n$ with Hilb poly P we have

① $H^i(I_Z(r)) = 0 \quad \forall \; i > 0$. and $r \geq m$.

② The graded module $\bigoplus_{r \geq m} H^0(I_Z(r)) \subset$ ~~sym~~ $k[x_0, \ldots, x_n]$

is generated in deg m.

Thus, by the uniform m-lemma we get a map $\underline{\mathrm{Hilb}}_X^P \to Gr$ which is injective. (at least on the level of k-points.).

Before we proceed, let us see if we have a natural transformtion of functors $\underline{\mathrm{Hilb}}_X^P \to Gr$. Let T be a scheme, and

$Z \subset \mathbb{P}^n_T$ an object of $\underline{\mathrm{Hilb}}_X^P(T)$.

From this, we want an object of $Gr(Q(m), \overbrace{\mathrm{sym}^m(k^{n+1})}^{V})$, i.e. a sub-vector-bundle of $V \times \mathcal{O}_T$. From the pointwise description, we know that this vector bundle should have fiber $H^0(I_{Z_t}(m))$ over $t \in T$. So we guess that this must be

$\pi_* I_Z(m)$, where $\pi: Z \to T$ is the projection.

From $0 \to I_Z(m) \to \mathcal{O}_{\mathbb{P}^n_T}(m) \to \mathcal{O}_Z(m) \to 0$ we have

$0 \to \pi_*(I_Z(m)) \to V_m \otimes \mathcal{O}_T \to \pi_* \mathcal{O}_Z(m) \to 0$

Fiberwise constant ranks but are they vector bundles!?

Questions we face :-

- Are $\pi_* I_Z(m)$ and $\pi_* O_Z(m)$ vector bundles ?

(We know that $H^0(I_{Z_t}(m))$ and $H^0(O_{Z_t}(m))$ have fixed rank for all $t \in T$, but ~~th~~ is this enough to guarranttee that π_* are locally free ?)

This raises an important technical issue :-

What is the relationship between the sheaf $\pi_* \mathcal{F}$ and the various $H^0(\mathcal{F}_t)$ as $t \in T$? (or $R^i\pi_*\mathcal{F}$ and $H^i(\mathcal{F}_t)$)

<u>Content</u> : Cohomology and base Change.

We'll see that in our case the π_* do turn out to be vector bundles and hence ~~gr~~ we get a natural transformation

$$\underline{\mathrm{Hilb}}_X^P \to Gr.$$

Finally, we must show that this is a locally closed embedding. That is, ~~at~~ ~~some~~ some how we must characterized the image by ~~vanis~~ equations. Where do these equations come from ?

Consider a point $S \subset V_m$ of Gr. When does S come from an ideal $\mathcal{I}_Z$, where $Z \subset \mathbb{P}^n$ has hilb poly P ?

We can set I = ideal gen. by S in $k[x_0, \ldots, x_n]$. But it will define a subscheme of Hilb poly P iff its r^{th} graded piece has the correct rank. ~~i.e.~~ ~~##~~ (for $r >> 0$). That is, the mult. maps

$$S \otimes \mathrm{Sym}^r \langle x_0, \ldots, x_n \rangle \longrightarrow \mathrm{Sym}^{m+r} \langle x_0, \ldots, x_n \rangle.$$

must have a prescribed <u>rank</u>. defined by minors !.

these give the equations.

More formally, we ~~let~~ consider the universal sequence

$$0 \to S \to V_m \otimes \mathcal{O}_{Gr} \to Q \to 0$$

And let $Z \subset Gr \times \mathbb{P}^n$ be the subscheme cut out by S.

Now Z/Gr will not be flat with Hilb poly P (not all S give the right hilb poly!).

but our Hilb is obtained by using the following –

<u>Thm</u>: There is a finite collection of polynomials $P_1, \dots, P_\ell$ and locally closed subschemes $H_1, \dots, H_\ell \subset Gr$ s.t.

$$Z|_{H_i} \to H_i \text{ is flat with Hilb poly } P_i \text{ and}$$

the stratification satisfies the universal property that if $\varphi: T \to Gr$ is a map s.t. $Z_T \to T$ is flat with Hilb poly P_i then φ factors through $H_i \hookrightarrow Gr$.

Our <u>Hilb</u>^{P} is one such stratum.

(In the proof, these H_i will be cut out by vanishing / non-vanishing of minors.).

<u>Thm's name</u>: Flattening Stratification.