Algebraic Curves and Surfaces I: Moduli of Curves

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Consider a reasonable class of objects in algebraic geometry. What constitutes 'reasonable' is hard to say in general, but it can include such disparate examples as

1. smooth compact curves,
2. subvarieties of a projective space,
3. maps between two varieties,
4. line bundles on a given variety,
5. branched covers of a given variety,

and so on. A remarkable feature of algebraic geometry is that the set of such objects is more than just a set—it is itself an algebraic variety, called the *moduli space* of those objects. The realization that algebro-geometric objects can be assembled into a moduli space goes back to Riemann, who suggested that isomorphism classes of smooth curves of genus g can be described as the points of a $3g - 3$ dimensional space. Making precise the idea of moduli spaces and building up the machinary to construct them took over a century. Now moduli spaces occupy a central place in algebraic geometry.

In this course, we will construct and study the moduli space of curves, the first example in the above list. Along the way, we will meet many other related moduli spaces. In fact, a large part of the course will develop ideas that are foundational to the study of moduli spaces in general; the space of curves will serve as a guide to not lead us astray.

Prerequisites: I will assume familiarity with the modern foundations of algebraic geometry (for example, [Har77, Chapter 2 and 3]), and a basic understanding of the theory of curves (an introductory course about algebraic curves or the knowledge of [Har77, Chapter 4] or [ACGH85, Chapter 1] should suffice).

Course plan

A rough plan follows. It is very much in flux, and likely to change depending on time, interests, necessary background and so on. Two good general references are *Moduli of Curves* by Harris and Morrison [HM98] and *Geometry of Algebraic Curves, Volume II* by Arbarello, Cornalba, and Griffiths [ACG11]. I will provide more detailed references for specific topics during the course.

1. **The Hilbert scheme**: The construction of the Hilbert scheme. Building on the Hilbert scheme to construct other moduli schemes such as Maps and Isoms. The local and global properties of the Hilbert scheme. Examples.

2. $\mathscr{M}_g$ **as a Deligne–Mumford stack**: The moduli problem of curves and the need for a new category of spaces. The definition of algebraic stacks. The construction of the moduli space $\mathscr{M}_g$ as a Deligne–Mumford stack.

3. **The Deligne–Mumford compactification** $\overline{\mathscr{M}}_g$: The stable reduction theorem and the Deligne–Mumford compactification of $\mathscr{M}_g$.

4. **Geometric Invariant Theory**: A brief foray into Geometric Invariant Theory (GIT). An outline of the GIT construction of $\overline{M}_g$.

5. **Elementary properties of** $\overline{\mathscr{M}}_g$: Basic deformation theory through deformations of nodal curves. Hurwitz's classical argument for the connectedness of $\overline{\mathscr{M}}_g$. The boundary stratification.

6. **Divisors on** $\overline{\mathscr{M}}_g$: The Picard group. The canonical bundle. Ample and effective divisors and the F-conjecture.

7. **The Kodaira dimension of** $\overline{\mathscr{M}}_g$: An outline of the Harris–Mumford–Eisenbud theorem that $\overline{M}_g$ is of general type for $g \geq 23$.

8. **Special topics** (Depending on time and interest, possibly with student participation): The cohomology and tautological rings. The Kontsevich space of maps, Gromov–Witten theory, and quantum cohomology. Hurwitz spaces and Severi varieties. Complete subvarieties of $\mathscr{M}_g$. The geometry of low genus cases, especially $\overline{M}_{0,n}$. A glimpse of the moduli of higher dimensional varieties.

References

[ACG11] Enrico Arbarello, Maurizio Cornalba, and Phillip A. Griffiths, *Geometry of algebraic curves. Volume II*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 268, Springer, Heidelberg, 2011, With a contribution by Joseph Daniel Harris. MR 2807457 (2012e:14059)

[ACGH85] Enrico Arbarello, Maurizio Cornalba, Phillip A. Griffiths, and Joe Harris, *Geometry of algebraic curves. Volume I*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 267, Springer-Verlag, New York, 1985. MR 770932 (86h:14019)

[Har77] Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52. MR 0463157 (57 #3116)

[HM98] Joe Harris and Ian Morrison, *Moduli of curves*, Graduate Texts in Mathematics, vol. 187, Springer-Verlag, New York, 1998. MR 1631825 (99g:14031)