Midterm exam questions

1. - Describe the points of Spec $\mathbb{C}[x,y]/(xy)$.
   - What are the stalks of the structure sheaf at the points (x,y), $(x-1,y)$, and (y)?
   - Which of these stalks are integral domains?
   - For which of these stalks is the natural map from $\mathbb{C}[x,y]/xy$ injective?
   - What is the closure of the point (y)?

2. Let $f\colon A \to B$ be a map of rings.
   - Prove that the preimage of a prime ideal is a prime ideal.
   - Is the preimage of a maximal ideal a maximal ideal?
   - Is the answer "yes" under some hypotheses on A and B?
   - Use the above to prove that, under the same hypotheses on A, the nilradical of A is the intersection of all maximal ideals of A.
   - Can you state the previous result in terms of functions on Spec A?
   - Take $A = \mathbb{C}[x]$ and $B = \mathbb{C}[x,y]/(y^2 - x)$ with the obvious map $f\colon A \to B$. Describe the sets Spec A, Spec B and the map between them induced by f.

3. Let A be a ring.
   - What is a distinguished open subset of Spec A?
   - Prove that the distinguished open subsets form a base of the Zariski topology.
   - Is every open subset a distinguished open subset?
   - Is the intersection of two distinguished open subsets a distinguished open subset?
   - Use the distinguished open subsets to show that Spec A is quasi-compact.

4. - Define a scheme.
   - Define an affine scheme.
   - Give an example of a scheme which is not an affine scheme.
   - Let X be the one pointed topological space. Let O_X be the sheaf of rings on X defined by $O_X(X) = \mathbb{Z}$. Is this a scheme?

5. Let $A = \mathbb{C}[x,y]/(xy)$ and $B = \mathbb{C}[x,y]/(xy^2)$.
   - Are Spec A and Spec B isomorphic as schemes?
   - What is the relationship between the underlying topological spaces?
   - Draw pictures of Spec A and Spec B.
   - There are (non-zero) $f \in A$ and $g \in B$ such that A_f and B_g are isomorphic. Using your pictures (or otherwise), find them.

6. Let $A = \mathbb{C}[t]$ and $P = \text{Proj } A[X,Y]$.
   - What are the points of P?

- What is $O_X(D_{XY})$?
- Let $Z \subset P$ be the closed subscheme cut out by $Y^2 - tX^2$. What are the points of Z and where are they mapped by the map $Z \to \text{Spec } A$?
- write Z as the Proj of a graded algebra.
- write down an affine open cover of Z.
- Is Z affine?