WHY ARE POINTS OF AN AFFINE SCHEME THE PRIME IDEALS?

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Back in the day, an algebraic variety used to be the set of solutions to a system of equations. That was good. Now, a scheme is the set of prime ideals of a ring. This is strange, and perhaps you don't like it (appeals to "the right definition" be damned!).

I want to convince you that a scheme is still, at heart, solutions to a system of equations, **and** it is also the set of prime ideals of a ring. They are two sides of the same coin.

Let us keep a running example $R = \mathbb{Q}[x, y]/(x^2 + y^2 - 1)$, although what I say will work for any ring. You should think of R as encoding the equation $x^2 + y^2 - 1$ with $\mathbb{Q}$ coefficients. A solution of this system is a choice of x and y that satisfy $x^2 + y^2 - 1$. But where should x and y live? Because $\mathbb{Q}$ is not algebraically closed, we miss a lot of solutions if we insist that they live in $\mathbb{Q}$. We could make them lie in $\overline{\mathbb{Q}}$—this is what people did in the 19th century—or we could not prescribe where they live. Let it be a free choice.

Let L be any field. An L-valued solution of our system is a choice of $x, y \in L$ that makes $x^2+y^2-1 = 0$. Equivalently (sanity check!), it is a ring homomorphism $R \to L$. So the points of the scheme spec R should be ring homomorphisms $R \to L$, where L is any field.

This is not so far from the truth. There is a small issue that we need to fix. Take $x = \frac{1}{\sqrt{2}}, y = \frac{1}{\sqrt{2}}$. It is an $\mathbb{R}$-valued solution of $x^2 + y^2 - 1 = 0$. It is also a $\mathbb{C}$-valued solution. And also a $\mathbb{Q}(\sqrt{2})$-valued solution. As our definition stands, these count as different solutions, but they should (obviously) be identified.

So, call $s_1 \colon R \to L_1$ and $s_2 \colon R \to L_2$ equivalent if there is a map $i \colon L_1 \to L_2$ such that $s_2 = i \circ s_1$. (This is not an equivalence relation, but just take the equivalence relation generated by this rule). The points of the scheme spec R are the equivalence classes. Done!

It turns out that your new and beautiful spec R is just the set of prime ideals of R. Here is how. Take a solution $s \colon R \to L$. I'll give you the corresponding prime ideal: it is $p = \ker s$. (Caution: the map s may not be surjective, so p has no reason to be maximal.) The assignment $s \rightsquigarrow p$ clearly (sanity check!) respects the equivalence relation. Going back, give me a prime ideal $p \subset R$. I'll give you an s: it is just the composite $R \to R/p \to \text{frac } R/p$. (It is not hard to check that this particular choice of s from the equivalence class is the "initial choice". Every other equivalent solution factors through this s). In this way, points of spec R are still just the solutions to a system of equations.

PS: From this POV, we can think about $r \in R$ as a function on points as follows. How do we evaluate r at an L-valued solution $s \colon R \to L$? We just take the image $s(r)$. The result lies in L.

PPS: A ring like $R = \mathbb{Z}$ is not a "system of equations" in the traditional sense, but our definition of an L-valued "solution" as a ring homomorphism and the equivalence with prime ideals still makes sense!

PPPS: Why must solutions take values in a field, and not an arbitrary ring? It turns out that you **can** expand your notion of solution to allow arbitrary rings, and there are advantages of doing this. As we saw, values in fields gives us the underlying set spec R. With a little work, we can also define the Zariski topology. But fields cannot detect non-reduced structure, so the data of the structure sheaf has to be tacked on. If you allow solutions in any ring, then you get enough information to recover also the structure sheaf.